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Analyticity results in Bernoulli Percolation

Agelos Georgakopoulos[1] and Christoforos Panagiotis[2]

[1,2]Mathematics Institute, University of Warwick, CV4 7AL, UK*,

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Abstract

We prove (rigorously) that in 2-dimensional Bernoulli percolation, the percolation density is an analytic function of the parameter in the supercritical interval. For this we introduce some techniques that have further implications. In particular, we prove that the susceptibility is analytic in the subcritical interval for all transitive short- or long-range models, and that $p_c^{bond} < 1/2$ for certain families of triangulations for which Benjamini & Schramm conjectured that $p_c^{site} \leq 1/2$.

1 Introduction

We prove that several functions studied in percolation theory are analytic functions of the percolation parameter. We consider Bernoulli bond percolation on a variety of graphs, as well as general long-range models (defined in Section 4.2) preserved by a transitive group action. The *susceptibility* χ of a percolation model is the expected number of vertices in the cluster of a fixed vertex o. The *percolation density* $\theta = \theta_o(p)$ is the probability that the cluster $C(o)$ of o is infinite. Let

$$p_{\mathbb{C}} := \inf_{p<1} \theta_o(p) \text{ is analytic in } (p, 1] \tag{1}$$

Our main results are

(i) For every quasi-transitive graph, and every quasi-transitive (1-parameter) long-range model, the susceptibility $\chi(p)$ is analytic in the subcritical interval $[0, p_c)$.

(ii) For every quasi-transitive lattice in $\mathbb{R}^2$, the (Bernoulli, bond) percolation density $\theta(p)$ is analytic in the supercritical interval $(p_c, 1]$ (in other words, $p_{\mathbb{C}} = p_c$). So is the n-point function τ and its truncation τ^f. The corresponding results are proved for continuum percolation in $\mathbb{R}^2$ as well.

(iii) For the infinite d-ary tree, we have $p_{\mathbb{C}} = p_c (= \frac{1}{d-1})$.

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(iv) For every finitely presented, 1-ended Cayley graph, we have $p_{\mathbb{C}} < 1$. Moreover, every finitely presented, 1-ended group has a Cayley graph with $p_c \leq p_{\mathbb{C}} \leq 1/2$ for both site and bond percolation.

(v) For every non-amenable graph with bounded degrees, we have $p_{\mathbb{C}} < 1$. It is possible for θ to be analytic at the uniqueness threshold p_u.

(vi) For certain families of triangulations for which Benjamini & Schramm [13] and Benjamini [12] conjectured that $p_c^{site} \leq 1/2$, we prove $p_c^{bond} \leq p_{\mathbb{C}} < 1/2$.

Most of percolation theory is concerned with phase transitions, and so a lot of its earlier work focused on the smoothness of functions like χ and θ that describe the macroscopic behaviour of its clusters. Kunz & Souillard [40] proved that the average number of clusters per site κ (the analogue of the free energy in the Ising model) is analytic for small enough p, and has a singularity at p_c for percolation on $\mathbb{Z}^d, d > 1$. Grimmett [26] proved that κ is C^∞ for $d = 2$. A breakthrough was made by Kesten [37], who proved that κ and χ are analytic for $p \in [0, p_c)$ for all d.[1] Smoothness results are harder to obtain in the supercritical interval $(p_c, 1]$, partly because the cluster size distribution $P_n := \mathbb{P}_t(|C(o)| = m)$ has an exponential tail below p_c (Section 3.2) but not above p_c [3]. Still, it is known that θ, κ, and the 'truncation' χ^f of χ are infinitely differentiable for $p \in (p_c, 1]$ on $\mathbb{Z}^d$ (see [19] or [28, §8.7] and references therein). It is a well-known open problem, dating back to [37] at least, and appearing in several textbooks ([38, Problem 6],[31, 28]), whether θ is analytic for $p \in (p_c, 1]$. Partial progress was made by Braga et.al. [16, 15], who showed that θ is analytic for p close enough to 1. In this paper we fully answer this question in the affirmative in the 2-dimensional case (Theorems 7.1 and 7.9). We also answer the corresponding question, asked by Günter et. al. [33], for the Boolean model (Theorem 9.1).

Part of the interest for this question comes form Griffiths' [30] discovery of models, constructed by applying the Ising model on 2-dimensional percolation clusters, in which the free energy is infinitely differentiable but not analytic. This phenomenon is since called a *Griffiths singularity*, see [54] for an overview and further references.

Kesten's method for the analyticity of χ (or κ) [37] (see also [28, §6.4]) involves extending p and χ to the complex plane, and applying the standard complex analytic machinery of Weierstrass to the series $\chi(p) := \sum_{n \in \mathbb{N}} n P_n(p)$. This uses the fact that $P_n(p)$ can be expressed as a polynomial by considering all possible clusters of size n, and can hence be extended to $\mathbb{C}$. To show that this series converges to an analytic function $\chi(z)$, one needs upper bounds for $|P_n(z)|$ inside appropriate domains in order to apply the Weierstrass M-test (see Appendix 15). These bounds are obtained combining the well-known fact due to Aizenman & Barsky [2] that $P_n(z)$ decays exponentially in n for real z, with elementary complex-analytic calculations. Kesten's calculations involved the numbers of certain 'lattice animals', but we observe (Theorem 4.11) that this is not necessary and his proof can be simplified. An immediate benefit of this simplification is that the proof extends beyond $\mathbb{Z}^d$, to bond and site percolation on any quasi-transitive graph. The only ingredients needed are the appropriate

[1] The threshold p_T in Kesten's original formulation was later shown to coincide with p_c by Aizenman & Barsky [2].

exponential decay statement and elementary complex analysis. Moreover, with a bit more work the proof can be extended to long-range models: the functions $P_n(p)$ are no longer polynomials, but we show (Theorem 4.8) how they can be extended into entire functions, i.e. complex-analytic functions defined for all $p \in \mathbb{C}$. This summarises the proof of (i), which is given in detail in Section 4.3. One application of (i) of particular interest to us is to a long-range model studied in [29], which was the original motivation of our work.

The technique we just sketched is used in our results (ii)–(v) as well, but additional ingredients are needed. For (ii), we write $\theta(p) = 1 - \sum_n P_n(p)$ by the definitions, but as P_n decays slower than exponentially for $p > p_c$ [40, 28], the above machinery cannot be applied to this series. Therefore, instead of working with the size of $C(o)$, we work with the 'perimeter' of its boundary. To make this more precise, define the *outer-interface* of the cluster $C(o)$ to be the pair $(\partial_{int}C(o), \partial_{ext}C(o))$, where $\partial_{int}C(o)$ denotes the set of edges of $C(o)$ bounding its outer face, and $\partial_{ext}C(o)$ denotes the set of vacant edges incident with the outside of $\partial_{int}C(o)$ (Figure 1). We say that such a pair of edge sets $I = (\partial_{int}C(o), \partial_{ext}C(o))$ *occurs* in some percolation instance, if it is the outer-interface of some cluster, in which case all edges in $\partial_{int}C(o)$ are occupied and all edges in $\partial_{ext}C(o)$ are vacant. For any plausible such I, the probability $P_I(p) := \mathbb{P}_p(I \text{ occurs})$ is just $p^{|\partial_{int}C(o)|}(1-p)^{|\partial_{ext}C(o)|}$ by the definitions, which is a polynomial we can extend to $\mathbb{C}$ hoping to apply our machinery. Moreover, these P_I exhibit the kind of exponential decay we need: $\partial_{ext}C(o)$ gives rise to a connected subgraph of the dual lattice, and we can combine a well-known coupling between supercritical bond percolation on a lattice and subcritical bond percolation on its dual (see Theorem 7.2) with the aforementioned exponential decay of P_n.

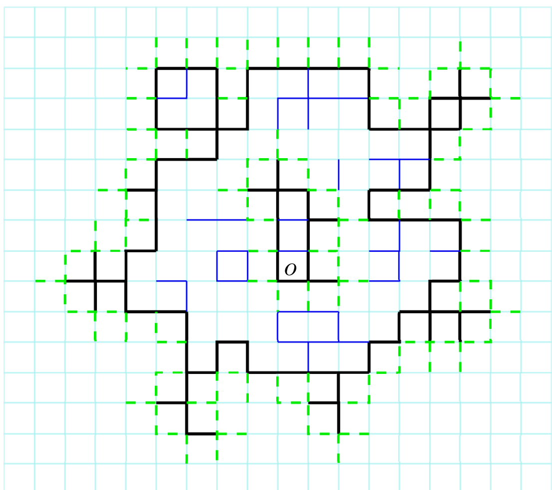

Figure 1: An example of two outer-interfaces of percolation clusters, one nested inside the other. We depict $\partial_{int}C(o)$ with bold lines, and $\partial_{ext}C(o)$ with dashed lines. The rest of the clusters is depicted in plain lines (blue, if colour is shown).

Still, further challenges arise when trying to express θ in terms of the functions P_I, because knowing that a certain outer-interface I occurs does not imply that it is part of the cluster $C(o)$: there could be other outer-interfaces nested inside I, as exemplified in Figure 1. We overcome this difficulty using the Inclusion-Exclusion Principle, to express θ as

$$\theta(p) = 1 - \sum_{I \in \mathcal{MS}} (-1)^{c(I)+1} P_I, \tag{2}$$

where $\mathcal{MS}$ is the set of finite disjoint unions of outer-interfaces, and $c(I)$ counts the number of outer-interfaces in I. The problem now becomes whether the probability for such an $I \in \mathcal{MS}$ with n edges in total decays exponentially in n. All we know so far is that the probability to have an outer-interface containing a fixed vertex x decays exponentially, which seems to be of little use given that there are many ways to partition n into smaller integers $n_1, \ldots n_k$, and construct an $I \in \mathcal{MS}$ out of k outer-interfaces of lengths n_i, each rooted at one of many candidate vertices x_i. But there is a way to bring all these possibilities under control, and establish the desired exponential decay, by a certain combination of the following ingredients:

a) the Hardy–Ramanujan formula (Section 3.4), implying that the number of partitions of an integer n grows subexponentially;

b) some combinatorial arguments that restrict the possible vertices x_i at which the outer-interfaces meet the horizontal axis, and

c) using the BK inequality (Theorem 3.2) to argue that for each choice of a partition of n, and vertices $x_1, \ldots x_k$, the probability of occurence of an $I \in \mathcal{MS}$ complying with this data decays as fast as if we had a single outer-interface of size n (which we already know to decay exponentially).

This summarises the proof of (ii), which is given in detail in Section 7. Our method applies to *site* percolation on the triangular lattice (Corollary 7.9) as well, but not to general 2-dimensional site percolation. Our proof does not provide enough evidence in order to conjecture that $p_{\mathbb{C}} = p_c$ for $\mathbb{Z}^d, d > 2$. The size distribution of the analog of an outer-interface (see below) is shown in [39] to decay slower than exponentially for $p < 1 - p_c$, hence the point $1 - p_c$ is a good candidate where to look for a singularity; see Section 10.6 for more. In a follow-up paper in preparation, we show that $p_{\mathbb{C}} \leq 1 - p_c$ for certain lattices in $\mathbb{Z}^d, d > 2$.

The only use of planarity in the proof of (ii) we just sketched was the duality argument needed for the exponential tail of the size of an outer-interface. It is easy to imagine generalising outer-interfaces to higher dimensions, although coming up with a precise definition that uniquely associates an interface with any cluster requires some thought. In Section 10 we offer such a definition that applies to all graphs, not just lattices in $\mathbb{R}^d$. We show that once we fix a 1-ended graph G, and a basis of its cycle space (for $G = \mathbb{Z}^d$ the family of all squares is a natural choice), every finite subgraph (aka. cluster) C of G uniquely defines a 'outer-interface' $I = (\partial_{int} C(o), \partial_{ext} C(o))$ with $\partial_{int} C(o)$ a connected subgraph of C, and $\partial_{ext} C(o)$ containing the minimal cut separating C from infinity. This refines the argument of Timar [51] used to simplify the proof of the theorem of Babson & Benjamini that $p_c < 1$ for every finitely presented Cayley graph. When G is such a Cayley graph, we show that our outer-interfaces exhibit an exponential tail by repeating the arguments (a)-(c) from above, and reach (iv) which is the deepest result of this paper (Theorem 10.12), mainly due to the 'deterministic' Theorem 10.4. This result also applies to site percolation (Corollary 10.15). Moreover, we show that if we 'triangulate' our Cayley graph by adding more generators, then we can achieve $p_c \leq p_{\mathbb{C}} \leq 1/2$ for both site and bond percolation (Theorem 10.16).

We remark that formula (2) can be thought of as a refinement of the well-known Peierls argument (see e.g. [28, p. 16]), where instead of an inequality we now have an equality. The price to pay is that the structures arising —of the form $(\partial_{int}C(o), \partial_{ext}C(o))$ instead of just $\partial_{ext}C(o)$— are harder to enumerate, and the benefit is that the events we consider are mutually exclusive, hence the equality. We found this technique very convenient in this paper and expect it to be useful elsewhere.

A well-known theorem of Benjamini & Schramm [13] states that $p_c(G) \leq \frac{1}{1+h_E(G)}$ where $h_E(G)$ is the Cheeger constant of an arbitrary graph G, and so $p_c < 1$ for every non-amenable graph. We show in Section 5, were we recall the relevant definitions, that the same bound applies to $p_{\mathbb{C}}$ (Theorem 5.1). This has the interesting consequence that θ does not witness the phase transition at the uniqueness threshold p_u: using the results of [44, 50], we deduce that θ is analytic at p_u for some Cayley graph of every non-amenable group. Another consequence of $p_c(G) \leq \frac{1}{1+h_E(G)}$ is (iii) (Corollary 6.1).

Most of this paper is concerned with analyticity results, but some of the methods developed can be applied to provide bounds on p_c as well. We display this in Section 11, where we prove that $p_c^{bond} < 1/2$ for certain families of triangulations for which Benjamini & Schramm [13], Benjamini [12], and Angel, Benjamini & Horesh [6] conjectured that $p_c^{site} \leq 1/2$ ((vi)).

After proving some functions to be analytic, additional fun, and hopefully results, can be had by studying their complex extensions. As already mentioned, we proved that the functions P_n admit entire extensions (trivially for nearest-neighbour models), and are therefore uniquely determined by their Maclaurin coefficients. As most observables of percolation theory, e.g. χ and θ, are uniquely determined by the sequence $\{P_n\}_{n \in \mathbb{N}}$, it makes sense to study those coefficients. We do so in Section 12, where we show that their signs alternate with n, and do not depend on the model (Theorem 12.1).

We use this fact in Section 13, where we show how one can make sense of a negative percolation threshold $p_c^- \in \mathbb{R}_{<0}$. As it happened in the history of p_c, more than one candidate definitions are possible. We could show that some of them coincide (Theorem 13.3), but there are still more questions than results on this topic.

Most of the essence of our proofs lies in combinatorial arguments. We have made an effort to make this paper accessible to the non-expert, except for this introduction that uses terminology that is defined later. The complex analysis we use is at undergraduate level, involving only some classics we recall in Appendix 15 and elementary manipulations. Hardly any background in probability theory is assumed, but some familiarity with the basics of percolation theory as in [28] will be helpful.

2 The setup

We recall some standard definitions of percolation theory in order to fix our notation. For more details the reader can consult e.g. [28, 41]. For a higher level overview of percolation theory we recommend the recent survey [20].

2.1 nearest-neighbour models

Let $G = (V, E)$ be a countably infinite graph, and let $\Omega := \{0,1\}^E$ be the set of *percolation instances* on G. We say that an edge e is *vacant* (respectively, *occupied*) in a percolation instance $\omega \in \Omega$, if $\omega(e) = 0$ (resp. $\omega(e) = 1$).

By (Bernoulli, bond) *percolation* on G with parameter $p \in [0,1]$ we mean the random subgraph of G obtained by keeping each edge with probability p and deleting it with probability $1-p$, with these decisions being independent of each other.

(More formally, we endow Ω with the σ-algebra $\mathcal{F}$ generated by the cylinder sets $C_e := \{\omega \in \Omega, \omega(e) = \epsilon\}_{e\in E, \epsilon\in\{0,1\}}$, and the probability measure defined as the product measure $\mathbb{P}_p := \Pi_{e\in E}\mu_e$, where $p \in [0,1]$ is our *percolation parameter* and μ_e is the Bernoulli measure on $\{0,1\}$ determined by $\mu_e(1) = p$.)

The *percolation threshold* $p_c(G)$ is defined by

$$p_c(G) := \sup\{p \mid \mathbb{P}_p(|C(o)| = \infty) = 0\},$$

where the *cluster* $C(o)$ of $o \in V$ is the component of o in the subgraph of G spanned by the occupied edges. It is well-known that $p_c(G)$ does not depend on the choice of o.

To define *site percolation* we repeat the same definitions, except that we now let $\Omega := \{0,1\}^V$, and let $C(o)$ be the component of o in the subgraph of G induced by the occupied vertices.

In this paper the graph G is a-priori arbitrary. Some of our results will need assumptions on G like vertex-transitivity or planarity, but these will be explicitly stated as needed.

2.2 long-range models

Long range percolation is a generalisation of Bernoulli bond percolation where different edges become occupied with different probabilities, and each vertex can have infinitely many incident edges that can become occupied. In fact, the graph is often taken to be the complete graph on countably many vertices, and so its edges play a rather trivial role. Therefore, it is simpler to define our model with a set rather than a graph as follows.

Let V be a countably infinite set (the *vertices*), and let $E = V^2$ be the set of pairs of its elements (the *edges*). We will typically write xy istead of $\{x,y\}$ to denote an element of E. Let $\mu : E \to \mathbb{R}_{\geq 0}$ be a function satisfying $\sum_{y\in V}\mu(xy) = 1$ for every $x \in V$ (in some occasions we allow more general μ, satisfying just $\sum_{y\in V}\mu(xy) < \infty$). The data V, μ define a random graph on V similarly to the previous definition, except that we now make each edge xy vacant with probability $e^{-\mu(xy)t}$, with our parameter t now ranging in $[0,\infty)$. The corresponding probability measure on $\Omega = \{0,1\}^E$ is denoted by $\mathbb{P}_t$ (We like thinking of t as time, with the each edge xy becoming occupied if vacant at a tick of a Poisson clock with rate $\mu(xy)$.)

Analogously to p_c, one defines

$$t_c = t_c(V,\mu) := \sup\{t \mid \mathbb{P}_t(|C(o)| = \infty) = 0\},$$

which again does not depend on the choice of $o \in V$.

We say that such a percolation model, defined by V and μ, is *transitive*, if there is a group acting transitively on V that preserves μ. In other words, if for every $x, y \in V$ there is a bijection $\pi : V \to V$ such that $\mu(\pi(z)\pi(w)) = \mu(zw)$.

Long range percolation is a less standard topic that is not typically found in textbooks, and the term often refers to the special case where the group acting transitively is $\mathbb{Z}$, for example in order to come up with a model in which θ is discontinous at t_c [5]. In the generality we work with it has been considered in e.g. [4, 29].

3 Definitions and preliminaries

3.1 Graph theoretic definitions

Let $G = (V, E)$ be a graph. An *induced* subgraph H of G is a subgraph that contains all edges xy of G with $x, y \in V(H)$. Note that H is uniquely determined by its vertex set. The subgraph of G *spanned* by a vertex set $S \subseteq V(G)$ is the induced subgraph of G with vertex set S.

The vertex set of a graph G will be denoted by $V(G)$, and its edge set by $E(G)$. A graph G is *(vertex-)transitive*, if for every $x, y \in V(G)$ there is an automorphism π of G mapping x to y, where an *automorphism* is a bijection π of $V(G)$ that preserves edges and non-edges.

A *planar graph* G is a graph that can be embedded in the plane $\mathbb{R}^2$, i.e. it can be drawn in such a way that no edges cross each other. Such an embedding is called a *planar embedding* of the graph. A *plane graph* is a (planar) graph endowed with a fixed planar embedding.

A plane graph divides the plane into regions called *faces*. Using the faces of a plane graph G we define its *dual graph* G^* as follows. The vertices of G^* are the faces of G, and we connect two vertices of G^* with an edge whenever the corresponding faces of G share an edge. Thus there is a bijection $e \mapsto e^*$ from $E(G)$ to $E(G^*)$.

3.2 Exponential tail of the subcritical cluster size distribution: the Aizenman-Barsky property

An important fact that will be used throughout the paper whenever we want to show the convergence of a series is the following exponential decay of the cluster size distribution $p_n := \mathbb{P}(|C(o)| = n)$ (or equivalently, of $f_n := \mathbb{P}(|C(o)| \geq n)$) in the subcritical regime, which we will call the *Aizenman-Barsky property*:[2]

Theorem 3.1 ([4, Proposition 5.1], [2, 7])**.** *For every quasi-transitive bond, site, or long-range model, (and any vertex o), if $p < p_c$ then*

$$\mathbb{P}_p(|C(o)| \geq n) = O(e^{-n/5\chi^2}).$$

[2]Some bibliographical remarks about Theorem 3.1: Kesten [37] proved exponential decay when $\chi < \infty$ for lattices in $\mathbb{R}^d$, and Aizenman & Newman [4] extended it to all models we are interested in (their precise formula is $\mathbb{P}_p(|C(o)| \geq m) \leq (e/m)^{1/2} e^{-m/(2\chi(p))^2}$). Aizenman & Barsky [2] proved $\chi < \infty$ below p_c on $\mathbb{Z}^d$, and [41, Theorem 7.46.] claims that 'their proof works in greater generality', that is, for all transitive graphs. Menshikov [43] independently obtained the same result in a more restricted class of models. Antunović & Veselić [7] extended this to all quasi-transitive models. Duminil-Copin & Tassion [22] gave a shorter proof that $\chi < \infty$ below p_c (or β_c) for all independent, transitive bond and site models.

3.3 The BK inequality

We define a partial order on our space $\Omega = \{0,1\}^{E(G)}$ of percolation instances as follows. For two configurations ω and ω' we write $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for every $e \in E$.

A random variable X is called *increasing* if whenever $\omega \leq \omega'$, then $X(\omega) \leq X(\omega')$. An event A is called *increasing* if its indicator function is increasing. For instance, the event $\{|C(o)| \geq m\}$ is increasing, where $C(o)$ as usual denotes the cluster of o.

For every $\omega \in \Omega$ and a subset $S \subset E$ we write

$$[\omega]_S = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for every } e \in S\}.$$

Let A and B be two events depending on a finite set of edges F. Then the disjoint occurrence of A and B is defined as

$$A \circ B = \{\omega \in \Omega : \text{ there is } S \subset F \text{ with } [\omega]_S \subset A \text{ and } [\omega]_{F\setminus S} \subset B\}.$$

Theorem 3.2. *(BK inequality)[53, 28] Let F be a finite set and $\omega = \{0,1\}^F$. For all increasing events A and B on Ω we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

3.4 Partitions of integers

A *partition* of a positive integer n is a multiset $\{m_1, m_2, \ldots, m_k\}$ of positive integers such that $m_1 + m_2 + \ldots + m_k = n$. Let $p(n)$ denote the number of partitions of n. An asymptotic expression for $p(n)$ was given by Hardy & Ramanujan in their famous paper [27]. An elementary proof of this formula up to a multiplicative constant was given by Erdős [23]. As customary we use $A \sim B$ to denote the relation $A/B \to 1$ as $n \to \infty$.

Theorem 3.3 (Hardy-Ramanujan formula)**.** *The number $p(n)$ of partitions of n satisfies*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The above asymptotic formula for $p(n)$ implies in particular that $p(n)$ grows subexponentially, and this is all we will need in our several applications of Theorem 3.3. This weaker statement can be proved much more easily, and we offer the following elementary proof that makes our paper more self-contained.

Lemma 3.4. *Let $p(n)$ denote the number of partitions of n. Then*

$$\limsup_{n\to\infty} p(n)^{1/n} = 1.$$

Proof. Let us denote $f(z)$ the generating function of $p(n)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n.$$

It is well known that $f(z) = \prod_{k=1}^{\infty} \dfrac{1}{1-z^k}$ (this follows easily by considering the bijection between the set of partitions of n and the set of sequences $(i_1, i_2, \ldots, i_n)$ where the i_j's are non-negative integers such that $i_1 + 2i_2 + \cdots + ni_n = n$).

The radius of convergence R of f is given by the formula

$$R = \frac{1}{\limsup_{n\to\infty} p(n)^{1/n}}.$$

It suffices to prove that $R = 1$. Since $f(1) = +\infty$ we have that $R \leq 1$. In order to show that $R \geq 1$ we will prove that f is analytic on the open unit disk.

Assume that $z \in [0, 1)$. Taking the logarithm of the infinite product we obtain the infinite sum $\sum_{k=1}^{\infty} -\log(1 - z^k)$. Using the fact

$$\lim_{x\to 0} \frac{-log(1 - x)}{x} = 1$$

and the convergence of the sum $\sum_{k=1}^{\infty} z^k$ we deduce that $\sum_{k=1}^{\infty} -\log(1 - z^k)$ converges. It follows that $\prod_{k=1}^{\infty} \frac{1}{1 - z^k}$, and hence $\sum_{n=0}^{\infty} p(n)z^n$, converges for every $z \in [0, 1)$. This in turn implies that $\sum_{n=0}^{\infty} p(n)z^n$ converges for every z in the open unit disk and thus it defines an analytic function. $\square$

4 The basic technique

A common ingredient of our analyticity results is the following technique, the main idea of which is present in [37] and was mentioned in the introduction. We express our function $f(p)$ as an infinite series $f(p) = \sum_{n\in\mathbb{N}} a_n f_n(p)$, where $f_n(p)$ is the probability of an event. For example, when $f = \chi$ is the expected size of the cluster $C(o)$ of o, then f_n is the probability that $|C(o)| = n$, and $a_n = n$. To prove that $f(p)$ is analytic, our strategy is to extend the domain of definition of each f_n to complex values of p (we will usually write z instead of p when doing so). Our extended f_n will turn out to be complex-analytic, and so f is analytic if the series $\sum_{n\in\mathbb{N}} a_n f_n(p)$ converges uniformly by standard complex analysis (Weierstrass' Theorem 15.1). To show the latter, we employ the Weierstrass M-test (Theorem 15.2), using upper bounds on $|f_n(z)|$ inside appropriate discs (centered in the interval $[0, 1]$ where p takes its values). These upper bounds are obtained by Lemma 4.1 below for nearest-neighbour models, and by its counterpart Lemma 4.4 for long-range models.

4.1 Nearest-neighbour models

The following lemma, and its generalisation Corollary 4.3 below, provides the upper bounds that we are going to plug into the M-test as explained above.

Let $\mathbb{P}_p$ denote the law of Bernoulli percolation with parameter p on an arbitrary graph G, as defined in Section 2. Let $D(x, M)$ denote the disc with center $x \in \mathbb{C}$ and radius $M \in \mathbb{R}_+$ in $\mathbb{C}$. For a subgraph S of G, let ∂S be the set of edges of G that have at least one end-vertex in S but are not contained in $E(S)$.

In this lemma, x is to be thought of as a value of our parameter p near which we want to show the analyticity of some function, and we are free to chose the radius M of the disc we consider as small as we like.

Lemma 4.1. *For every finite subgraph S of G and every $o \in V(G)$, the function $P(p) := \mathbb{P}_p(C(o) = S)$ admits an entire extension $P(z), z \in \mathbb{C}$, such that for* *every* $1 > M > 0$*, every* $1 > x \geq 0$ *with* $x + M < 1$ *and every* $z \in D(x, M)$*, we have*

$$|P(z)| \leq c^{|\partial S|} P(x + M),$$

where $c = c_{M,x} := \frac{1-x+M}{1-x-M}$*.*

Moreover, $|P(z)| \leq c_M^{|\partial S|} P(1-M)$ *for every* $z \in D(1, M)$*, where* $c_M := \frac{1+M}{1-M}$*.*

(The second sentence will be used to prove analyticity at $p = 1$; the reader who is only interested in analyticity for $p \in [0, 1)$ may ignore it and skip the last paragraph of the proof.)

Proof. By the definitions, we have

$$P(p) = (1-p)^{|\partial S|} p^{|E(S)|}, \tag{3}$$

because the event $\{C(o) = S\}$ is satisfied exactly when all edges in ∂S are absent and all edges in $E(S)$ present. This function, being a polynomial, admits an entire extension, which we will still denote by $P = P(z)$ with a slight abuse.

To prove the upper bound in our first statement —for $1 > x \geq 0$, and $z \in D(x, M)$— we will bound each of the two products appearing in (3) separately. Easily,

$$|z|^{|E(S)|} \leq (x + M)^{|E(S)|}$$

when $z \in D(x, M)$ because $|z| \leq x + |z - x| \leq x + M$.

Moreover, it is geometrically obvious that the distance $|1 - z|$ between 1 and z, is maximised at $z = x - M$, which implies

$$|1 - z|^{|\partial S|} \leq (1 - x + M)^{|\partial S|}.$$

Plugging these two inequalities into (3) we obtain the desired inequality:

$$|P(z)| \leq (1 - x + M)^{|\partial S|}(x + M)^{|E(S)|} =$$

$$\Big(\frac{1 - x + M}{1 - x - M}\Big)^{|\partial S|}(1 - x - M)^{|\partial S|}(x + M)^{|E(S)|} = \Big(\frac{1 - x + M}{1 - x - M}\Big)^{|\partial S|} P(x + M),$$

where we also applied (3) with $p = x + M$.

For the second statement, let $z \in D(1, M)$. Then $|z| \leq 1 + M$, and $|1 - z| \leq M$, and similarly to the above calculation we have

$$|P(z)| \leq M^{|\partial S|}(1 + M)^{|E(S)|} =$$

$$M^{|\partial S|}\Big(\frac{1 + M}{1 - M}\Big)^{|E(S)|}(1 - M)^{|E(S)|} = \Big(\frac{1 + M}{1 - M}\Big)^{|E(S)|} P(1 - M).$$

□

Remark 4.2. *When* G *has maximum degree* d*, we have the crude bound* $|\partial S| \leq d|S|$*, with which Lemma* (4.1) *yields* $|P(z)| \leq c_{M,x}^{d|S|} P(x + M)$*.*

Note that in the proof of Lemma 4.1 we can replace $E(S)$ and ∂S with any two disjoint finite sets of edges $D, F \subset E(G)$, to obtain the following:

Corollary 4.3. *For every two disjoint finite sets of edges* $D, F \subset E(G)$*, the function* $P(p) := \mathbb{P}_p(D \subseteq \omega$ *and* $F \cap \omega = \emptyset)$ *(i.e. the probability that all edges in* D *are occupied and all edges in* F *are vacant) admits an entire extension* $P(z), z \in \mathbb{C}$*, such that* $|P(z)| \leq \Big(\frac{1-x+M}{1-x-M}\Big)^{|F|} P(x + M)$ *for every* $M > 0$*,* $1 > x \geq 0$ *with* $x + M < 1$ *and* $z \in D(x, M)$*.* □

4.2 Long-range models

We now prove the analogue of Lemma 4.1 for long-range models. Recall that in our long-range setup, we have a vertex set V and any two of its elements can form an edge. The parameters x, M now take their values in $[0, \infty)$, as this is the case for our percolation parameter t. Let ∂S be the set of pairs $\{x, y\} \subset V^2$ that are not contained in $E(S)$ but have at least one vertex in S.

Lemma 4.4. *For every finite graph S on a subset of V, and every $o \in V$, the function $P(t) := \mathbb{P}_t(C(o) = S)$ admits an entire extension $P(z), z \in \mathbb{C}$, such that $|P(z)| \leq e^{2M|S|}P(x + M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x, M)$.*

The proof of this is similar to that of Lemma 4.1, but as our function $P(t)$ is not exactly a polynomial now we will need some reshuffling of terms and the following basic fact about complex numbers.

Proposition 4.5. *For every $\mu > 0$ and every $z \in \mathbb{C}$ we have*

$$|e^{\mu z} - 1| \leq e^{\mu|z|} - 1.$$

Proof. Expressing $e^{\mu z}$ via its Maclaurin expansion and using the triangle inequality yields

$$|e^{\mu z} - 1| = \left| \sum_{j=1}^{\infty} \frac{(\mu z)^j}{j!} \right| \leq \sum_{j=1}^{\infty} \frac{|z\mu|^j}{j!}. \tag{4}$$

Since $\mu > 0$, the last expression coincides with the Maclaurin expansion of $e^{\mu r} - 1$ evaluated at $r = |z|$, from which we obtain $|e^{\mu z} - 1| \leq e^{\mu|z|} - 1$.

□

Proof of Lemma 4.4. Similarly to (3), we have

$$\mathbb{P}_t(C(o) = S) = \prod_{e \in \partial S} e^{-t\mu(e)} \prod_{e \in E(S)} \left(1 - e^{-t\mu(e)}\right), \tag{5}$$

because the event $\{C(o) = S\}$ is satisfied exactly when all edges in ∂S are absent and all edges in $E(S)$ present. Multiplying the second product by $\prod_{e \in E(S)} e^{t\mu(e)}$ and the first by its inverse, we obtain

$$\mathbb{P}_t(C(o) = S) = \prod_{e \in \partial S \cup E(S)} e^{-t\mu(e)} \prod_{e \in E(S)} \left(e^{t\mu(e)} - 1\right) = e^{-t\mu(S)} \prod_{e \in E(S)} \left(e^{t\mu(e)} - 1\right), \tag{6}$$

where $\mu(S) := \sum_{e \text{ incident with } S} \mu(e)$ because the edges incident with S are exactly the elements of $\partial S \cup E(S)$. This function clearly admits an entire extension, which we will still denote by $P = P(z)$ with a slight abuse.

To prove the upper bound, we will bound each of the two products appearing in (6) separately. Easily,

$$|e^{-z\mu(S)}| \leq e^{2M|S|} e^{-(x+M)\mu(S)}$$

when $z \in D(x, M)$ because $|z| \leq x + |z - x| \leq x + M$ and $\mu(S) \leq |S|$. For the second product, we apply Proposition 4.5 to each factor to obtain

$$|e^{z\mu(e)} - 1| \leq e^{|z|\mu(e)} - 1 \leq e^{(x+M)\mu(e)} - 1 \tag{7}$$

for every for $z \in D(x,M)$.

Combining these two inequalities, and then applying (6) with $t = x + M$, we obtain the desired bound:

$$|P(z)| \leq e^{2M|S|} e^{-(x+M)\mu(S)} \prod_{e \in E(S)} \left(e^{(x+M)\mu(e)} - 1\right) = e^{2M|S|} P(x+M).$$

$\square$

Again, in this proof we can replace $E(S)$ and ∂S with any two disjoint finite sets of edges $D, F \subset E$, to obtain, in analogy with Corollary 4.3, the following statement:

Corollary 4.6. *For every two disjoint finite sets of edges $D, F \subset E$, the function $P(t) := \mathbb{P}_t(D \subseteq \omega$ and $F \cap \omega = \emptyset)$ (i.e. the probability that all edges in D are occupied and all edges in F are vacant) admits an entire extension $P(z), z \in \mathbb{C}$, such that $|P(z)| \leq e^{2M|V(D \cup F)|} P(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$, where $V(D \cup F)$ denotes the set of vertices that are incident with some edge in $D \cup F$.* $\square$

Similarly, if we replace $E(S)$ in Lemma 4.4 with a set of edges incident to a vertex o and ∂S with the remaining edges that are incident to o we obtain the following corollary. We let $N(o)$ denote the neighbourhood of o in the percolation cluster, i.e. the set of vertices sharing an occupied edge with o.

Corollary 4.7. *For every $o \in V$ and every $L \subset V$, the function $P(t) := \mathbb{P}_t(N(o) = L)$ admits an entire extension $P(z), z \in \mathbb{C}$, such that $|P(z)| \leq e^{2M} P(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$.*

4.2.1 Analyticity of the probability of a given cluster size

Next, we prove that $p_m(t) := \mathbb{P}_t(|C(o)| = m)$ is analytic, in the full generality of our long-range models as above. For nearest-neighbour models this is trivial, because the corresponding probability can be expressed as a polynomial, but the long-range variant is more interesting. In addition to analyticity, the following result also provides the upper bound that we will plug into the Weirstrass M-test to deduce the analyticity of the susceptibility χ for subcritical long-range models (Theorem 4.11).

Theorem 4.8. *For every $m \in \mathbb{N}$ and every $o \in V$, the function $p_m(t) := \mathbb{P}_t(|C(o)| = m)$ admits an entire extension $p_m(z), z \in \mathbb{C}$, such that $|p_m(z)| \leq e^{2Mm} p_m(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$.*

Proof. For $m \in \mathbb{N}$, let $\mathcal{G}_m(V)$ denote the set of finite graphs whose vertex set in a subset of V with m elements containing o (to be thought of as possible percolation clusters of o). For every such $S \in G_m(V)$, Lemma 4.4 yields an entire extension P_S of $\mathbb{P}_t(C(o) = S)$. We claim that the sum

$$\sum_{S \in G_m(V)} P_S(z), \tag{8}$$

which for $z \in \mathbb{R}, z > 0$ coincides with $\mathbb{P}_z(|C(o)| = m)$, converges uniformly on each closed disc $D(x,M), M > 0$, $x \geq 0$ to a function $p_m : \mathbb{C} \to \mathbb{C}$. By Weierstrass' Theorem 15.1, this means that p_m admits an entire extension.

Indeed, this uniform convergence follows from the Weierstrass M-test: each summand P_S can be bounded by $|P_S(z)| \leq e^{2M|S|}P_S(x+M) = e^{2Mm}P_S(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x, M)$ by Lemma 4.4. Moreover, the sum of these bounds satisfies

$$\sum_{S \in G_m(V)} e^{2Mm} P_S(x + M) = e^{2Mm} p_m(x + M) < \infty.$$

Thus the Weierstrass M-test can be applied to deduce that (8) converges uniformly on $D(x, M)$, and therefore on any compact subset of $\mathbb{C}$.

Finally, the above bounds also prove that $|p_m(z)| \leq e^{2Mm} p_m(x + M)$ as desired. □

Corollary 4.9. *For every $m \in \mathbb{N}$ and every $o \in V$, the function $f_m(t) := \mathbb{P}_t(|C(o)| \geq m)$ admits an entire extension.*

Proof. It follows from the formula $\mathbb{P}_t(|C(o)| \geq m) = 1 - \sum_{i=1}^{m-1} \mathbb{P}_t(|C(o)| = i)$ and Theorem 4.8. □

4.3 Analyticity of χ in the subcritical regime

In this section we prove that the *susceptibility* $\chi(t) := \mathbb{E}_t(|C(o)|)$ of our models is an analytic function of the parameter in the subcritical interval. This applies to both nearest-neighbour and long-range models. For this we need to assume that our model has the Aizenman-Barsky property.

Theorem 4.10. *For every long-range model with the Aizenman-Barsky property (in particular, for every transitive model), $\chi(t)$ is real-analytic in the interval $[0, t_c)$.*

Theorem 4.11. *For every bounded-degree nearest-neighbour model with the Aizenman-Barsky property (in particular, for every vertex-transitive graph), $\chi(p)$ is real-analytic in the interval $[0, p_c)$.*

The proofs of these facts are very similar, and follow Kesten's proof [37] of the corresponding statement for (nearest-neighbour) lattices in Z^d, except that we simplify it by avoiding any mention to lattice animals.

Proof of Theorem 4.10. Each summand in the definition $\chi(t) = \sum_{m=1}^{\infty} m\mathbb{P}_t(|C(o)| = m)$ of χ admits an analytic extension to $\mathbb{C}$ by Theorem 4.8. By Weierstrass' Theorem 15.1, it suffices to prove that for every $x \in [0, t_c)$ there is an open disk D centred at x such that $\sum_{m=1}^{\infty} m\mathbb{P}_t(|C(o)| = m)$ converges uniformly in D.

Pick an arbitrary $x \in [0, t_c)$ and $x < y < t_c$. It is proved in [2] that $\chi(t) < \infty$ for every $t < t_c$, and so we have $\chi := \chi(y) < \infty$.

Since we are assuming the Aizenman-Barsky property, we have $\mathbb{P}_y(|C(o)| \geq m) \leq e^{-m/5\chi^2}$. Since $p_m(t) := \mathbb{P}_t(|C(o)| \geq m)$ is an increasing function of t, we deduce

$$p_m(t) \leq e^{-m/5\chi^2} \tag{9}$$

for every $t \leq y$. Pick $M > 0$ small enough that $x + M \leq y$ and $e^{2M}e^{-1/5\chi^2} < 1$, that is, $M < \min\{y - x, \frac{1}{10\chi^2}\}$. Combined with Theorem 4.8, this implies that

$|p_m(z)| \leq Ca^m$ for $z \in D(x, M)$, where C is a positive constant and $a < 1$. Since $\sum_{m=1}^{\infty} Cma^m < \infty$, we can use the Weierstrass M-test to conclude that the sum $\sum_{m=1}^{\infty} mp_m(z)$ converges uniformly on $D(x, M)$ and since each p_m is analytic the sum is also analytic. Moreover, this sum coincides with $\chi(t)$ for $t \in D(x, M) \cap [0, t_c)$, and so our statement follows. $\square$

Proof of Theorem 4.11. This is similar to the above, but instead of Theorem 4.8 we use the corresponding statement for nearest-neighbour models. This is easier, as the sum (8) is finite. Applying Lemma 4.1 (using the bounded degree assumption, see also Remark 4.2) yields an upper bound of the form $|p_m(z)| \leq c^{dm} P(x + M)$ which we use instead of that of Theorem 4.8 in our application of the M-test. The rest of the proof is identical to that of Theorem 4.10. $\square$

The above proofs show that there is an open disk centred at any subcritical value x of the parameter where p_m converges exponentially fast to 0. Easily, every higher moment $\mathbb{E}(|C(o)|^k) = \sum_{m=1}^{\infty} m^k \mathbb{P}_t(|C(o)| = m)$ (or for the same reason, the expectation of every sub-exponential function of $|C(o)|$) admits an analytic extension on the same disk, and so we obtain

Corollary 4.12. *Every moment $\mathbb{E}_x(|C(o)|^k)$ is an analytic function of the parameter x in the subcritical interval for all models as in Theorem 4.11 or Theorem 4.10.*

Let us summarize the ideas used to prove the analyticity of χ. Our proofs had little to do with χ itself. The main idea was to express χ as a sum of multiples of probabilities of events, namely $\chi(t) = \sum_{m=1}^{\infty} m\mathbb{P}_t(|C(o)| = m)$, and use the exponential decay of those probabilities (Theorem 3.1) to counter the exponential growth of their complex extensions (as in Lemma 4.1) in small enough discs around every point p. The rest of the proof was standard complex analysis, namely the Weierstrass M-test and Weierstrass' Theorem 15.1thmWei. As we are going to use the same proof structure several times, we reformulate it as the following corollary, which is a straightforward generalisation of the proof of Theorem 4.11. To formulate it, we need the following definition.

Definition 4.13. *We say that an event E —of a nearest-neighbour model on a graph G— has* complexity *n, if it is a disjoint union of a family of events $(F_n)_{n \in \mathbb{N}}$ where each F_i is measurable with respect to a set of edges of G of cardinality n.*

Corollary 4.14. *Let $\mathbb{P}_p$ denote the law of a nearest-neighbour model, and let $f(p)$ be a function that can be expressed as $f(p) = \sum_{n \in \mathbb{N}} \sum_{i \in L_n} a_i \mathbb{P}_p(E_{n,i})$ in an interval $p \in I \subset [0, 1]$, where $a_n \in \mathbb{R}$, L_n is a finite index set, and each $E_{n,i}$ is an event measurable with respect to $\mathbb{P}_p$ (in particular, the above sum converges absolutely for every $p \in I$). Suppose that*

(i) $E_{n,i}$ has complexity of order $\Theta(n)$, and

(ii) for each open subinterval $J \subset I$ there is a constant $0 < c_J < 1$ such that $\sum_{i \in L_n} a_i \mathbb{P}_p(E_{n,i}) = O({c_J}^n)$.

Then $f(p)$ is analytic in I.

Proof. We imitate the proof of Theorem 4.10, except that instead of the Aizenman-Barsky property we use our assumption (ii), and instead of Lemma 4.1 we use its generalisation Corollary 4.3, which we apply to the sequence of events witnessing that $(E_{n,i})$ satisfies (i). (Note that the complexity of an event governs the exponential growth rate of the maximum modulus of the extension of its probability to a complex disc as a function of the radius of that disc.) $\square$

Remark: A similar statement for long-range models can be formulated, and proved, along the same lines, except that we use the total μ-weight rather than the cardinality of an edge-set in Definition 4.13.

5 $p_{\mathbb{C}} < 1$ for non-amenable graphs

The *(edge)-Cheeger constant* of a graph G is defined as $h_E(G) := \inf_S \frac{|\partial_E S|}{|S|}$, where the infimum ranges over all finite subgraphs S of G. When $h_E(G) > 0$ we say that G is *non-amenable*. A well-known theorem of Benjamini & Schramm [13] states that $p_c(G) \leq \frac{1}{1+h_E(G)}$. We show here that the same bound applies to $p_{\mathbb{C}}$. We use the same technique as in the subcritical case (Section 4.3), except that we replace the Aizenman-Barsky property with an observation of Pete that the arguments of Benjamini & Schramm imply the exponential decay above the aforementioned threshold of the 'truncated' cluster size for non-amenable graphs.

Theorem 5.1. *For every bounded degree graph G with $h := h_E(G) > 0$, we have $p_{\mathbb{C}} \leq \frac{1}{1+h_E(G)}$.*

Proof. By the definitions, we have $1 - \theta(p) = \sum_n \mathbb{P}_p(|C(o)| = n)$.

The statement follows if we can apply Corollary 4.14 for $I = (\frac{1}{1+h_E(G)}, 1]$ and $E_n := \{|C(o)| = n\}$ (and $a_n = 1$). So let us check that the assumptions of Corollary 4.14 are satisfied.

The exponential decay condition (ii) is established in [47, Proposition 12.9], which states that for every $p \in (\frac{1}{1+h_E(G)}, 1]$ we have $\mathbb{P}_p(|C(o)| = n) \leq \mathbb{P}_p(n \leq |C(o)| < \infty) < e^{-rn}$ for some constant $r = r(p) > 1$, and it is clear from the proof that $r(p)$ is monotone in p.

For condition (i), we note that if d is the maximum degree of G, then E_n has complexity at most dn, as it is the disjoint union of the events of the form $C(o) = S$ where S ranges over all connected subgraphs of G with n vertices containing o. We have thus proved that all assumptions of Corollary 4.14 are satisfied as claimed. $\square$

Remark: The same proof applies if we replace θ by some other subexponential funtion of the restriction of $|C(o)|$ to finite values.

It is well known that when G is amenable and transitive, there can never be more than one infinite cluster, whence $p_c = p_u$ [18] where

$$p_u = \inf\{p \in [0,1] : \text{ there exists a unique infinite cluster}\}.$$

On the other hand, Benjamini & Schramm [13] conjectured that $p_c < p_u$ holds on every non-amenable transitive graph.

It is natural to ask whether θ witnesses the phase transition at p_u whenever $p_c < p_u$, i.e. whether θ is non-analytic at p_u. It turns out that this is not the case, i.e. there are examples of Cayley graphs where θ is analytic at p_u. Indeed, Thom [50], refining the result of Pak & Smirnova-Nagnibeda [44], proved that whenever the spectral radius $\rho(G)$ of G is at most $1/2$ we have $p_c < p_u$. In fact, it follows from their proof that $p_u > \frac{1}{1+h_E(G)}$. Moreover they proved that $\rho(G) \leq 1/2$, and so $p_u > \frac{1}{1+h_E(G)}$, for some Cayley graph of any non-amenable group. (See [48] for other conditions that imply $p_u > \frac{1}{1+h_E(G)}$.) But then Theorem 5.1 yields that θ is analytic at p_u.

6 $p_{\mathbb{C}} = p_c$ for regular trees

It is well-known that if G is a d-regular tree for $d > 2$, then $p_c = \frac{1}{d-1}$ [41, 47], and it is easy to prove that $h_E(G) = d-2$ in this case. Thus Theorem 5.1 immediately yields

Corollary 6.1. *If T is the d-regular tree, then $p_{\mathbb{C}} = p_c = \frac{1}{d-1}$.*

For $d = 3$ this is rather trivial, since $\theta(p)$ can be computed exactly using a recursive formula: assuming the root o has degree $d-1$, and all other vertices degree d, we have $1-\theta' = (1-p\theta')^{d-1}$. From this it is easy to compute $\theta = \theta_o(p)$ when o has degree d as well: we have $1-\theta = (1-p\theta')^d$. For $d = 3$ we have $\theta'(p) = \frac{2p-1}{p^2}$ and hence $\theta(p) = 1-(1-\frac{2p-1}{p})^3$. We remark that this function is convex, corroborating the common belief about the shape of θ in general (see e.g. [28, p. 148]). The cases $d = 4, 5$ can also be solved exactly as they boil down to finding roots of polynomials of degree 3 and 4 respectively. For high values of d the Abel–Ruffini theorem kicks in, and Galois theory implies that our equation is in general not soluble in terms of radicals.

It was proved by Brillinger [17] (using the implicit function theorem) that each root of a polynomial is an analytic function of the coefficients of the polynomial in any interval in which no two roots coincide. We could deduce Corollary 6.1 from Brillinger's theorem if we knew that no roots of the above equation collide in the interval $p \in (p_c, 1]$. Checking whether this condition is satisfied can be done with a certain amount of elementary manipulations, and so Corollary 6.1 can indeed be deduced from Brillinger's theorem[3]. Our approach yields a more probabilistic approach which we find overall simpler.

We finish this section with an open problem:

Problem 6.1. *Does $p_{\mathbb{C}} = p_c$ hold for every tree (for which $p_c < 1$)?*

7 Analyticity above the threshold for planar lattices

A *quasi-transitive lattice* (in $\mathbb{R}^2$) is a connected plane graph L such that for some pair of linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$, translation by each v_i preserves L, and this action has finitely many orbits of vertices. *Remark:* The

[3]We thank Damiano Testa for some of these remarks.

seemingly more general definition as a plane graph admitting a semiregular action of the group $\mathbb{Z}^2$ (by isometries of $\mathbb{R}^2$ preserving L, or even more generaly by arbitrary graph-theoretic isomorphisms) with finitely many orbits of vertices can be proved to be in fact equivalent, but we will not go into the details; the main idea is to embed a fundamental domain of L with respect to that action in a square, and then tile $\mathbb{R}^2$ by copies of that square. Another approach can be found in [14, Proposition 2.1]. Theorem 7.1 does not apply to a lattice H in hyperbolic 2-space just because Theorem 7.2 below fails, but our proof shows that $p_{\mathbb{C}}(H) \leq 1 - p_c(H^*)$. In this case we have $1 - p_c(H^*) = p_u(H)$ by [14, Theorem 3.8][4]; in other words, we have shown analyticity of θ above p_u for all planar lattices.

In this section we prove

Theorem 7.1. *For Bernoulli bond percolation on any quasi-transitive lattice we have* $p_{\mathbb{C}} = p_c$.

This result is new even for the standard square lattice $\mathbb{Z}^2$, i.e. the Cayley graph of $\mathbb{Z}^2$ with respect to the standard generating set $\{(0,1),(1,0)\}$. Sligtly more effort is needed to prove it in the generality of quasi-transitive lattices. The reader that just wants to see a simplest possible proof for the lattice $L = \mathbb{Z}^2$ is advised to:

- ignore Theorem 7.2, and just recall that $p_c(\mathbb{Z}^2) = 1/2$ and $\mathbb{Z}^{2*} = \mathbb{Z}^2$;
- skip the definition of X in Section 7.1, and instead take X to be the horizontal 'axis' of $\mathbb{Z}^2$, and X^+ the right 'half-axis' starting at the origin o; and
- notice that Proposition 7.3 holds trivially with $f = 1$.

We will use the following important fact about the relation between the percolation thresholds in the primal and dual lattice. The history of this result starts with the Harris-Kesten theorem that $p_c(\mathbb{Z}^2) = 1/2$. A special case was obtained by Bollobas & Riordan [8], and almost simultaneously the general case was proved by Sheffield [49, Theorem 9.3.1] in a rather involved way. A shorter proof can be found in [21].

Theorem 7.2 ([49]; see also [21])**.** *For every quasi-transitive lattice L, we have* $p_c(L) + p_c(L^*) = 1$.

The analog of Theorem 7.1 for Bernoulli site percolation on the triangular lattice can be proved along the same lines, see Corollary 7.9. We do not use any notion of duality in this case, but it becomes important that $\dot{p}_c = 1/2$. Our proof does not apply to site percolation on arbitrary planar lattices.

7.1 Preliminaries on quasi-transitive lattices

We will construct a 2-way infinite path X in any quasi-transitive lattice L, which can be thought of as a 'quasi-geodesic' of both L and L^*. Alternatively, we could take X to be a 2-way infinite geodesic of L and prove Proposition 7.3

[4]The fact that non-amenability is equivalent to hyperbolicity in this setup is well-known, see e.g. [25].

below differently, but the approach we follow is not more complicated and has the advantage that it avoids the axiom of (countable) choice.

Since L is a plane graph, we naturally identify $V(L)$ with a set of points of $\mathbb{R}^2$. Let $o \in \mathbb{R}^2$ be a vertex of L and recall that $o + kv_1 \in V(L)$ for some non-zero vector $v_1 \in \mathbb{R}^2$ and every $k \in \mathbb{Z}$. Fix a path P from o to $o+v_1$. We may assume without loss of generality that P does not contain $o + kv_1$ for $k \neq 0, 1$, for otherwise we can replace v_1 by one of its multiples and P by a subpath. Note that the union $\bigcup_k P + kv_1$ of its translates along multiples of v_1 contains a 2-way infinite path X. Moreover, it is not too hard to see that we can choose X to be periodic, i.e. to satisfy $X + tv_1 = X$ for some $t \in \mathbb{N}$. For convenience, we will assume without loss of generality that o lies in X. Let X^+, X^- denote the two 1-way infinite sub-paths of X starting at o.

Proposition 7.3. *Let L be a quasi-transitive lattice and X^+ the o-∞ path defined above. Then there is a constant $f = f(L)$ such that every subgraph of L that surrounds o and has at most N edges must contain one of the first fN vertices X^+, and every subgraph of L^* that surrounds o and has at most N edges must contain (the dual of) one of the first fN edges of X^+.*

Proof. Suppose $S \subset L$ surrounds o. Then S must separate o from infinity, and so it must meet the infinite path X^+. Similarly, any $S \subset L^*$ surrounding o must cross both X^+ and X^-.

By quasi-transitivity, the lengths of the edges of L and L^* are bounded above by some $B \in \mathbb{N}$. Recall that X is periodic, and let D denote the diameter (with respect to the euclidean metric of $\mathbb{R}^2$) of a period of X. It follows easily that for some $f \leq D/L$

□

7.2 Main result

Let L be a quasi-transitive lattice, and o a vertex of L fixed throughout this section. A subgraph S of L is called a *outer-interface* (of o) if there is a finite connected subgraph H of L containing o such that S consists of the vertices and edges incident with the unbounded face of H.

The *boundary* ∂S of an outer-interface S is the set of edges of L that are incident with S and lie in the unbounded face of S. It is important to remember that ∂S may contain edges that have both their end-vertices in S; our proof will break down (at Lemma 7.5) if we exclude such edges from the definition of ∂S. Let $|S| := |E(S)|$ be the number of edges in S.

Given a realisation $\omega \in 2^{E(L)}$ of our Bernoulli percolation on L, we say that an outer-interface S *occurs* in ω if S is the boundary of the unbounded face of some cluster of ω. This happens exactly when all edges of S are occupied and all edges in ∂S are vacant.

The following is an easy consequence of the definitions.

Lemma 7.4. *If two occuring outer-interfaces share a vertex then they coincide.*

□

The following is one of the reasons why our proof only applies to lattices rather than arbitrary planar graphs.

Lemma 7.5. *For every outer-interface S we have $|\partial S| \geq |S|/k$ for some integer $k = k(L)$.*

For example, if L is the square lattice $\mathbb{Z}^2$, then $k = 2$. (And not $k = 1$, because it can happen that most edges in ∂S have both their end-vertices in S; for example, we can have a 'space filling' outer-interface whose vertex set is an $n \times n$ box of $\mathbb{Z}^2$. The following proof will give a worse bound than $k = 2$, but we can afford to be generous.)

Proof. We first observe that every face of L is bounded: choose vectors v_1, v_2 as in the definiton of a quasi-transitive lattice, fix a path P_1 from o to $o + v_1$ and a path P_2 from o to $o + v_2$, and note that the union of their translates $kP_i, k \in \mathbb{Z}$ contains a grid that separates $\mathbb{R}^2$ into bounded regions containing the faces of L.

Therefore, since there are finitely many of orbits of vertices of L, there are also finitely many of orbits of faces, and so the number of edges in the boundary of any face is at most some $k' \in \mathbb{N}$. Now recall that S consists of the vertices and edges incident with the unbounded face F of some $H \subset L$. If we walk along the boundary S of F, we will never traverse k' or more edges of S without encountering an edge in ∂S, and we will encounter each edge in ∂S at most twice. Thus our assertion holds for $k = 2(k' - 1)$. □

A *multi-interface* M is a finite set of pairwise vertex-disjoint outer-interfaces.

Lemma 7.6. *For every outer-interface S, the edge-set ∂S^* spans a connected subgraph of L^* surrounding o. Similarly, for every multi-interface M, the edge-set ∂M^* spans a subgraph of L^* the number of components of which equals the number of outer-interfaces in M (and each of these components surrounds o).*

Proof. Recall that S consists of the vertices and edges incident with the unbounded face F of a finite connected subgraph H in a fixed embedding of L. Let J be a Jordan curve disjoint from H such that H lies in the bounded side of J, and J is close enough to H that it meets all edges in ∂S and no other edges of L. Then the cyclic sequence of faces and edges of L visited by J defines a closed walk in L^*, proving that ∂S^* spans a connected subgraph of L^*. That this subgraph surrounds o is an immediate consequence of the definition of an outer-interface.

Now let M be a multi-interface comprising the outer-interfaces $S_1, \ldots, S_m$. We just proved that each ∂S_i^* spans a connected subgraph of L^*, so it only remains to show that ∂M^* contains no path between ∂S_i^* and ∂S_j^* for $i \neq j$. Since S_i and S_j are vertex-disjoint, one of them is contained in a bounded face of the other. Let us assume that S_i is contained in a bounded face of S_j. Then it is easy to see that the edges of S_i contain a cut of L^* separating ∂S_i^* from ∂S_j^*. Since ∂M^* contains no edge of S_i^* by the vertex-disjointness of the S_i, this proves our claim that ∂M^* contains no path between ∂S_i^* and ∂S_j^*. □

Let $\mathcal{MS}$ denote the set of multi-interfaces of L. We say that $M \in \mathcal{MS}$ *occurs* if each of the outer-interfaces it contains occurs. Let $|M| := \sum_{S_i \in M} |S_i|$ be the total number of edges in M. Let $\partial M := \bigcup_{S_i \in M} \partial S_i$, and let $\mathcal{MS}_n := \{M \in \mathcal{MS} \mid |\partial M| = n\}$ be the set of multi-interfaces with n boundary edges.

Lemma 7.7. *There is a constant $r \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ at most $r^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any percolation instance ω.*

Proof. Suppose $M \in \mathcal{MS}_n$ occurs in ω. Since occurring outer-interfaces meet X^+ by Proposition 7.3, and they are vertex-disjoint by Lemma 7.4, M is uniquely determined by the subset D of $\{x_0, x_1, \ldots\}$ it meets, in other words, $M = \bigcup_{x_i \in D} S(x_i, \omega)$, where $S(x_i, \omega)$ denotes the occuring outer-interface containing x_i.

Note that $|S(x_i, \omega)| > i/f$ for every $x_i \in D$ by Proposition 7.3. Since $kn \geq |M| = \sum_{x_i \in D} |S(x_i, \omega)|$ by Lemma 7.5 and the above remark, we deduce $fkn > \sum_{x_i \in D} i$. This means that D uniquely determines a partition of a number smaller than fkn. Moreover, distinct occurring multi-interfaces in $\mathcal{MS}_n$ determine distinct subsets D of $\{x_0, x_1, \ldots\}$, and therefore distinct partitions. By the Hardy–Ramanujan formula, the number of such partitions is less than $r^{\sqrt{n}}$ for some $r > 0$. Thus less than $r^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in ω. □

If $C(o)$ is finite, then there is exactly one outer-interface that occurs and is contained in $C(o)$, namely the boundary of the unbounded face of $C(o)$. We denote the probability of this event by P_S, that is, we set

$$P_S(p) := \mathbb{P}(S \text{ occurs and } S \subset C(o)).$$

Thus we can write the probability $\theta_o(p)$ that $C(o)$ is finite by summing P_S over all $S \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of outer-interfaces:

$$1 - \theta_o(p) = \textstyle\sum_{S \in \mathcal{S}} P_S(p) \tag{10}$$

for every $p \in (p_c, 1]$.

As usual, our strategy to prove the analyticity of θ, is to express θ as an infinite sum of functions that admit analytic extensions, namely, probabilities of events that depend on finitely many edges, and then apply Corollary 4.14. Formula (10) is a first step in this direction, however, the functions P_S are not fit for our purpose: the event $\{S \text{ occurs and } S \subset C(o)\}$ is not measurable with respect to the set of edges incident with S only. Therefore, we would prefer to express θ in terms of the simpler functions

$$Q_S := \mathbb{P}_p(S \text{ occurs}).$$

These functions have the advantage that comply with the premise of Corollary 4.3, and hence $|Q_S(p)|$ is bounded in $D(p, M)$ by $e^{C_{M,p}|S|} Q_S(p+M)$, where $C_{M,p}$ is independent of S. But when trying to write θ as a sum involving these Q_S, we have to be more careful: we have

$$1 - \theta_o(p) = \mathbb{P}_p(|C(o)| < \infty) = \mathbb{P}_p(\text{at least one } S \in \mathcal{S} \text{ occurs})$$

by the definitions, but more than one $S \in \mathcal{S}$ might occur simultaneously. Therefore, we will apply the inclusion-exclusion principle to the set of events $\{S \text{ occurs}\}_{S \in \mathcal{S}}$. We claim that

$$1 - \theta_o(p) = \textstyle\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} Q_M(p) \tag{11}$$

for every $p \in (p_c, 1]$, where $c(M)$ denotes the number of outer-interfaces in the multi-interface M.

To prove this, we need first of all to check that the sum in the right hand side converges. This is implied by Lemma 7.8 below, which states that

$\sum_{M\in\mathcal{MS}_n} Q_M(p)$ decays exponentially in n, and therefore our sum converges absolutely. Then, we need to check that this sum agrees with the inclusion-exclusion formula. This is so because, for every set I of outer-interfaces, we have $\mathbb{P}(\text{every } S \in I \text{ occurs}) = 0$ unless the elements of I are pairwise vertex-disjoint —that is, $I \in \mathcal{MS}$— by Lemma 7.4 and so we can restrict the inclusion-exclusion formula to $\mathcal{MS}$ rather than consider sets of outer-interfaces that intersect.

The main part of our proof is to show that the probability for at least one multi-interface in $\mathcal{MS}_n$ to occur decays exponentially in n, which will imply the following lemma. The rest of the arguments used to prove Theorem 7.1 are identical to those of e.g. Theorem 4.11 or 5.1.

Lemma 7.8. *For every $p \in (p_c, 1]$ there are constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$ with $c_2 < 1$, such that for every $n \in \mathbb{N}$,*

$$\sum_{M\in\mathcal{MS}_n} Q_M(p) \leq c_1 c_2{}^n. \tag{12}$$

Moreover, if $[a, b] \subset (p_c, 1]$, then the constants c_1 and c_2 can be chosen independent of p in such a way that (35) holds for every $p \in [a, b]$.

The proof of this is based on the fact that the size of the boundary of an outer-interface S that contains a certain vertex x has an exponential tail. This is because ∂S is contained in a component of the dual L^* by Lemma 7.6, and as our percolation is subcritical on L^*, the Aizenman-Barsky property holds. Still, the exponential tail of each $|\partial S|$ does not easily imply Lemma 7.8. First of all, the sum in the left hand side of Lemma 35 is larger than the probability $\mathbb{P}(\mathcal{MS}_n \text{ occurs})$ that a multi-interface of $\mathcal{MS}_n$ occurs. Second, a multi-interface might consist of plenty of outer-interfaces. Nevertheless, we will be able to overcome these difficulties. Using Lemma 7.7 we prove that the aforementioned sum does not grow too fast when compared with the probability that a multi-interface of $\mathcal{MS}_n$ occurs.

Proof of Lemma 7.8. We start by noticing that

$$\sum_{M\in\mathcal{MS}_n} Q_M(p) = \mathbb{E}_p\Big(\sum_{M\in\mathcal{MS}_n} \chi_{\{M \text{ occurs}\}}\Big),$$

where χ_A denotes the characteristic function of the occurence of an event A. The number of multi-interfaces $M \in \mathcal{MS}_n$ that can occur simultaneously is bounded above by $r^{\sqrt{n}}$ for some $r > 0$ by Lemma 7.7. It follows that

$$\sum_{M\in\mathcal{MS}_n} \chi_{\{M \text{ occurs}\}} \leq r^{\sqrt{n}} \chi_{\{\mathcal{MS}_n \text{ occurs}\}}$$

which in turn implies that

$$\sum_{M\in\mathcal{MS}_n} Q_M(p) \leq r^{\sqrt{n}} \mathbb{P}_p(\mathcal{MS}_n \text{ occurs}).$$

Hence it suffices to show that $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$ decays exponentially in n. In order to do so we will employ the exponential tail of the size of a certain (subcritical) cluster in the dual L^* given by the Aizenman-Barsky property. For this we will use the natural coupling of the percolation processes on L and L^*: given a percolation instance $\omega \in 2^{E(L)}$ on L, we obtain a percolation instance

ω^* on L^* by changing the state of each edge, i.e. letting $\omega^*(e^*) = 1 - \omega(e)$ for every $e \in E(L)$. Let $C(k)$ denote the event that there is a subgraph of a cluster of ω^* that surrounds o and contains at most k edges. Note that $C(k)$ is an increasing event. We claim that

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq \sum_{\{m_1,\ldots,m_k\} \in T_n} \mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)), \tag{13}$$

where $\circ$ means that the events occur edge–disjointly (see Section 3.3), and T_n is the set of partitions of n with the property that for every $N \leq n$ at most fN elements of the partition have size at most N, where f is the constant of Proposition 7.3. Once this claim is established, we will be able to employ the BK inequality (Theorem 3.2) to bound $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$.

To prove (13), we remark that each $M \in \mathcal{MS}_n$ defines a partition $\{m_1, \ldots, m_k\}$ of n by letting m_i stand for the number of edges in the ith component K_i of the subgraph of L^* spanned by ∂M^*. By Lemma 7.6 each K_i surrounds o, and so if M occurs then K_i is a witness of $C(m_i)$, and these witnesses are pairwise edge-disjoint. Thus the occurence of M implies the occurence of the event $C(m_1) \circ \ldots \circ C(m_k)$ in ω^*. To conclude that (13) holds, we apply the union bound to the family of events of the latter form, ranging over all partitions $\{m_1, \ldots, m_k\} \in T_n$, but we still need to check that for any $M \in \mathcal{MS}_n$ the corresponding partition lies in T_n. This is true by Proposition 7.3 and the pigeonhole principle, since the K_i are pairwise edge-disjoint by definition.

The BK inequality [28] states that

$$\mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)) \leq \mathbb{P}_{1-p}(C(m_1)) \cdot \ldots \cdot \mathbb{P}_{1-p}(C(m_k)).$$

Recall that a subgraph of L^* with at most m_i edges surrounding o must contain one of the first fm_i vertices of X by Proposition 7.3. Combining this fact with the union bound, and applying the Aizenman-Barsky property, we obtain $\mathbb{P}_{1-p}(C(m_i)) \leq fm_i c^{m_i}$ for some constant $0 < c = c(p) < 1$. As $\mathbb{P}_{1-p}(C(n)) < 1$ for every n, we deduce that $\mathbb{P}_{1-p}(C(m_i)) \leq (c+\varepsilon)^{m_i}$ for some $\varepsilon > 0$ such that $c + \varepsilon < 1$; indeed, for any ε, this is satisfied for large enough m_i, and raising ε we can make it true for the smaller values of m_i. In addition, if $[a, b] \subset (p_c, 1]$, then the monotonicity of $\mathbb{P}_{1-p}(C(m_i))$ implies that the constant $c + \varepsilon$ can be chosen uniformly for $p \in [a, b]$.

Combining all these inequalities starting with (13) we conclude that

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq |T_n|(c+\varepsilon)^n.$$

We have $T_n \leq h^{\sqrt{n}}$ for some constant h by the Hardy–Ramanujan formula (Theorem 3.3), and so

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq h^{\sqrt{n}}(c+\varepsilon)^n.$$

Thus $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$ decays exponentially in n as claimed. □

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. As already explained, Lemma 7.8 implies that the inclusion–exclusion expression (11) holds. The assertion follows if we can apply Corollary 4.14 for $I = (p_c, 1]$, $L_n = \mathcal{MS}_n$, and $(E_{n,i})$ an enumeration of the events

$\{M \text{ occurs}\}_{M\in\mathcal{MS}_n}$. So let us check that the assumptions of Corollary 4.14 are satisfied.

By definition, every $M \in \mathcal{MS}_n$ has n vacant edges. Moreover, $|M| \leq k(L)n$ by Lemma 7.5. Thus assumption (i) of Corollary 4.14 is satisfied. The fact that assumption (ii) is satisfied is exactly the statement of Lemma 7.8. □

Corollary 7.9. *For Bernoulli site percolation on the triangular lattice, the percolation density $\theta(p)$ is analytic for $p \in (1/2, 1]$.*

Proof. The proof is essentially the same as that of Theorem 7.1, with the simplifications proposed after the statement of that theorem for the case $L = \mathbb{Z}^2$ still applying. The only difference is that instead of the coupling with percolation on the dual L^* that we used there, which we combined with Theorem 7.2 to deduce the exponential decay of the probability that a fixed vertex lies on an outer-interface of length n, we now obtain this exponential decay by noticing that if S is an occuring outer-interface, then the vertices incident with ∂S that do not lie in S form a connected vacant subgraph. But vacant subgraphs are subcritical when $p > \dot{p}_c = 1/2$, and so applying the Aizenman-Barsky property to them yields the desired exponential decay. □

8 Analyticity of τ above p_c for planar lattices

In this section we prove that the n-point function τ and its truncated version τ^f are also analytic functions of p for $p > p_c(L)$, where as in the previous section L is any quasi-transitive lattice in $\mathbb{R}^2$.

Given a k-tuple $\boldsymbol{x} = \{x_1, \ldots, x_k\}, k \geq 2$ of vertices of L, the function $\tau_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a cluster of Bernoulli percolation on L with parameter p. Similarly, $\tau^f_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a *finite* cluster. The *diameter* $diam(\boldsymbol{x})$ of $\boldsymbol{x}$ is defined as $\max_{1\leq i<j\leq k}\{d(x_i, x_j)\}$ where $d(x_i, x_j)$ denotes the graph-theoretic distance between x_i and x_j.

Theorem 8.1. *For every quasi-transitive lattice L and every finite set $\boldsymbol{x}$ of vertices of L, the functions $\tau_{\boldsymbol{x}}(p)$ and $\tau^f_{\boldsymbol{x}}(p)$ admit analytic extensions to a domain of $\mathbb{C}$ that contains the interval $(1/2, 1]$. Moreover, for every $p \in (p_c, 1]$ there is a closed disk $D(p, \delta), \delta > 0$ and positive constants $c_1 = c_1(p, \delta), c_2 = c_2(p, \delta) < 1$ such that*

$$|\tau^f_{\boldsymbol{x}}(z)| \leq c_1 c_2^{diam(\boldsymbol{x})}$$

for every $z \in D(p, \delta)$ for such an analytic extension $\tau^f_{\boldsymbol{x}}(z)$ of $\tau^f_{\boldsymbol{x}}(p)$.

Proof. We start by showing that $\tau^f_{\boldsymbol{x}}(p)$ is analytic. Suppose $\boldsymbol{x} = \{x_1, \ldots, x_k\}$. Let $C_{x_i \not\to x_j}$ denote the event that there is an outer-interface that surrounds x_i but not x_j or vice versa[5]. It is easy to see that

$$\tau^f_{\boldsymbol{x}}(p) = \mathbb{P}_p(A) - \mathbb{P}_p(B),$$

where

$$A := \bigcup_{1\leq i\leq k} \{|C(x_i)| < \infty\}$$

[5]In this section the term outer-interface is used as in Section 7.2, except that we no longer require it to surround o.

and

$$B := \bigcup_{1\le i<j\le k} \{C_{x_i \not\leftrightarrow x_j}\}.$$

We will prove that both $\mathbb{P}_p(A)$ and $\mathbb{P}_p(B)$ are analytic functions of p in $(p_c, 1]$. Let $\mathcal{MS}(\boldsymbol{x})$ denote the set of multi-interfaces each element of which surrounds at least one element of $\boldsymbol{x}$ and $\mathcal{MS}'(\boldsymbol{x})$ the set of multi-interfaces each element of which surrounds some but not all $x_i \in \boldsymbol{x}$. By the definitions, $\mathcal{MS}'(\boldsymbol{x})$ is a subset of $\mathcal{MS}(\boldsymbol{x})$. Using the inclusion-exclusion principle we obtain

$$\mathbb{P}_p(A) = \sum_{M\in\mathcal{MS}(\boldsymbol{x})} (-1)^{c(M)+1}\mathbb{P}_p(M \text{ occurs})$$

and

$$\mathbb{P}_p(B) = \sum_{M\in\mathcal{MS}'(\boldsymbol{x})} (-1)^{c(M)+1}\mathbb{P}_p(M \text{ occurs}),$$

provided these sums converge, which will follow as for the corresponding formula (11) for θ. Moreover, provided that $\sum_{M\in\mathcal{MS}_n(\boldsymbol{x})}\mathbb{P}_p(M \text{ occurs})$ decays exponentially, the same holds for $\sum_{M\in\mathcal{MS}'_n(\boldsymbol{x})}\mathbb{P}_p(M \text{ occurs})$, where $\mathcal{MS}_n(\boldsymbol{x})$ and $\mathcal{MS}'_n(\boldsymbol{x})$ denote those multi-interfaces M of $\mathcal{MS}(\boldsymbol{x})$ and $\mathcal{MS}'(\boldsymbol{x})$, respectively, with $|\partial M| = n$. If this is the case, then the analyticity of $\mathbb{P}_p(A)$ and $\mathbb{P}_p(B)$ follows as in the proof of Theorem 7.1.

So let us show that $\sum_{M\in\mathcal{MS}_n(\boldsymbol{x})}\mathbb{P}_p(M \text{ occurs})$ decays exponentially in n. First of all, we need an upper bound for the number of multi-interfaces $M \in \mathcal{MS}_n(\boldsymbol{x})$ that can occur simultaneously. It is easy to see that a crude upper bound is $\tilde{s}(n)^k$, where $\tilde{s}(n)$ is equal to $s(1) + s(2) + \cdots + s(n)$ and $s(i)$ is the maximum number of multi-interfaces in $\mathcal{MS}_i(\{x\})$ for any $x \in \boldsymbol{x}$ that can occur simultaneously. This is true because a multi-interface $M \in \mathcal{MS}_n(\boldsymbol{x})$ comprises (possibly empty) multi-interfaces $M_1, \ldots, M_k$, where each element of M_i surrounds x_i, and each M_i has boundary of size at most n. In Lemma 7.7 we showed that $s(n)$ is bounded above by $r^{\sqrt{n}}$ for some $r > 0$,and so

$$\tilde{s}(n) \le \left(r^{\sqrt{1}} + r^{\sqrt{2}} + \cdots + r^{\sqrt{n}}\right) \le nr^{\sqrt{n}}.$$

Similarly to Lemma 7.8, we now obtain

$$\sum_{M\in\mathcal{MS}_n(\boldsymbol{x})} \mathbb{P}_p(M \text{ occurs}) \le n^k r^{k\sqrt{n}}\mathbb{P}_p(\mathcal{MS}_n(\boldsymbol{x}) \text{ occurs}).$$

Thus it suffices to show that $\mathbb{P}_p(\mathcal{MS}_n(\boldsymbol{x}) \text{ occurs})$ has an exponential tail.

Observe that M is the union of k (not necessarily disjoint) multi-interfaces $M_1, \ldots, M_k$ that surround $x_1, \ldots, x_k$ respectively. By the pigeonhole principle there is some i such that $|\partial M_i| \ge |\partial M|/k$. Using the union bound as in (13), except that we now also sum over all $x_i \in \boldsymbol{x}$, we obtain

$$\mathbb{P}_p(\mathcal{MS}_n(\boldsymbol{x}) \text{ occurs}) \le k \sum_{\{m_1,m_2,\ldots,m_k\}\in\tilde{T}_n} \mathbb{P}_{1-p}(C(m_1)\circ C(m_2)\circ\cdots\circ C(m_k)),$$

where $\tilde{T}_n$ is the number of partitions of some number $m \in \{\lceil n/k\rceil, \ldots, n\}$, and we only consider partitions with the property that for every $N \le m$ at most fN elements of the partition have size at most N, where again f is the constant of

Proposition 7.3. The exponential decay of $\mathbb{P}_p(\mathcal{MS}_n(\boldsymbol{x})$ occurs) follows as in the proof of Lemma 7.8. This completes the proof that $\tau^f_{\boldsymbol{x}}(p)$ is analytic in $(p_c, 1]$.

We proceed with $\tau_{\boldsymbol{x}}$. Since $\tau^f_{\boldsymbol{x}}$ is analytic, it suffices to show that $\tau_{\boldsymbol{x}} - \tau^f_{\boldsymbol{x}}$ is analytic. It is known that the infinite cluster is unique in our setup ([18, 47]), and this implies that $\tau_{\boldsymbol{x}} - \tau^f_{\boldsymbol{x}} = \mathbb{P}(|C_{x_1}| = \infty, \ldots, |C_{x_k}| = \infty)$. The latter probability is complementary to the probability $\mathbb{P}(A)$, which we have just shown to be analytic. Hence $\tau_{\boldsymbol{x}} - \tau^f_{\boldsymbol{x}}$ is analytic as well.

For the second claim of the theorem, using the inclusion-exclusion principle once more, we write

$$\tau^f_{\boldsymbol{x}}(p) = \textstyle\sum_{M \in \mathcal{MS}^{\forall}(\boldsymbol{x})} (-1)^{c(M)+1} \mathbb{P}_p(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\}), \tag{14}$$

where $\mathcal{MS}^{\forall}(\boldsymbol{x})$ is the set of multi-interfaces each element of which surrounds every element of $\boldsymbol{x}$, and $\{\boldsymbol{x}$ is connected$\}$ means there is a path from x_i to x_j for every i and j. The key observation is that the size of the boundary of every element of $\mathcal{MS}^{\forall}(\boldsymbol{x})$ is bounded below by $diam(\boldsymbol{x})/g$ for some constant g independent of $\boldsymbol{x}$. Indeed, this can be proved similarly to the proof of Proposition 7.3. Our goal is to prove that for every $p \in (p_c, 1]$ there is $\delta > 0$ such that $\mathbb{P}_z(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\})$ decays exponentially in the size of the boundary M for every $z \in D(p, \delta)$ (here,
$\mathbb{P}_z(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\})$ denotes the entire extension of
$\mathbb{P}_p(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\}), p \in [0, 1]$, which exists since the latter is a finite polynomial in p). Then the desired inequality will follow from (14).

Given a multi-interface $M \in \mathcal{MS}^{\forall}(\boldsymbol{x})$, consider the unique outer-interface M_0 of M which is surrounded by every other outer-interface of M. Since M_0 does not separate the elements of $\boldsymbol{x}$, it is easy to see that when M occurs, any occurring outer-interface C that surrounds some $x_i \in \boldsymbol{x}$ but not some other $x_j \in \boldsymbol{x}$ is *strictly contained* in the finite component K of the complement of ∂M_0. By 'strictly contained' we mean that both the open edges of C and its boundary are contained in K. Denoting $B(M)$ the event that there is no such outer-interface C for any pair of i and j, we notice that the events $B(M)$ and $\{M$ occurs$\}$ depend on the state of disjoint sets of edges. Hence they are independent and we obtain

$$\mathbb{P}_p(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\}) = \mathbb{P}_p(M \text{ occurs})\mathbb{P}_p(B(M)).$$

Using the inclusion exclusion principle we conclude that

$$\mathbb{P}_p(B(M)) = 1 + \sum_{S \in \mathcal{MS}(\boldsymbol{x}, M)} (-1)^{c(S)} \mathbb{P}_p(S \text{ occurs}),$$

where $\mathcal{MS}(\boldsymbol{x}, M)$ is the set of multi-interfaces comprising outer-interfaces each witnessing that $B(M)$ fails. The set $\mathcal{MS}(\boldsymbol{x}, M)$ is a subset of $\mathcal{MS}(\boldsymbol{x})$, which implies that for every $z \in \mathbb{C}$,

$$\sum_{S \in \mathcal{MS}(\boldsymbol{x}, M)} |\mathbb{P}_z(S \text{ occurs})| \leq \sum_{S \in \mathcal{MS}(\boldsymbol{x})} |\mathbb{P}_z(S \text{ occurs})|,$$

where as usual $\mathbb{P}_z(\cdot)$ denotes the entire extension of $\mathbb{P}_p(\cdot)$. We proved above that $\sum_{M \in \mathcal{MS}_n(\boldsymbol{x})} \mathbb{P}_p(M \text{ occurs})$ decays exponentially in n for every $p \in (p_c, 1]$.

By Corollary 4.3 and Lemma 7.5 we deduce that for every $p \in (p_c, 1]$ and $\delta > 0$ there is $c = c(p, \delta) > 0$ such that

$$|\mathbb{P}_z(S \text{ occurs})| \leq c(p, \delta)^n \mathbb{P}_{p'}(S \text{ occurs}),$$

where $p' = p + \delta$ when $p < 1$ and $p' = 1 - \delta$ when $p = 1$. (To prove analyticity at $p = 1$ we need to extend Corollary 4.3 to also provide a bound similar to that of the last sentence of Lemma 4.1, but this is straightforward.) Hence

$$\sum_{S \in \mathcal{MS}_n(\boldsymbol{x})} |P_z(S \text{ occurs})| \leq c(p, \delta)^n \sum_{S \in \mathcal{MS}_n(\boldsymbol{x})} P_{p'}(S \text{ occurs}),$$

and choosing δ small enough we obtain that

$$\sum_{S \in \mathcal{MS}_n(\boldsymbol{x})} |\mathbb{P}_z(S \text{ occurs})| \leq dc^n$$

holds for every $z \in D(p, \delta)$. Using the triangle inequality and summing over all n, we thus obtain

$$|\mathbb{P}_z(B(M))| \leq 1 + \frac{dc}{1 - c}$$

for every $z \in D(p, \delta)$.

Combining all the above we conclude that

$$|\tau_{\boldsymbol{x}}^f(z)| \leq \sum_{M \in \mathcal{MS}^{\forall}(\boldsymbol{x})} |\mathbb{P}_z(\{M \text{ occurs}\} \cap \{\boldsymbol{x} \text{ is connected}\})| \leq$$

$$\Big(1 + \frac{dc}{1 - c}\Big) \sum_{n=m'}^{\infty} \sum_{M \in \mathcal{MS}_n^{\forall}(\boldsymbol{x})} |\mathbb{P}_z(M \text{ occurs})| \leq \Big(1 + \frac{dc}{1 - c}\Big) \frac{dc^{m'}}{1 - c},$$

where $m' := diam(\boldsymbol{x})/g$ and as usual $\mathcal{MS}_n^{\forall}(\boldsymbol{x})$ denotes the set of those elements of $\mathcal{MS}^{\forall}(\boldsymbol{x})$ with boundary of size n. The proof is now complete. $\square$

Theorem 8.1 has the following corollary

Theorem 8.2. *For every $k \geq 1$, every quasi-transitive lattice L, and $o \in V(L)$, the functions $\chi_k^f(p) := \mathbb{E}_p(|C(o)|^k; |C(o)| < \infty)$ are analytic in p.*

Proof. Let us show that $\chi^f(p) := \mathbb{E}(|C(o)|; |C(o)| < \infty)$ is analytic. The case $k \geq 2$ will follow similarly. We observe that, by the definitions,

$$\chi^f(p) = \sum_{x \in V(L)} \tau^f(o, x) = 1 + \sum_{y \in V(L) \setminus \{o\}} \tau^f(o, y).$$

The probabilities $\tau^f(o, y)$ admit analytic extensions by Theorem 8.1, and so it suffices to prove that the sum $\sum_{y \in L \setminus \{o\}} \tau^f(o, y)$ converges uniformly on an open neighbourhood of $(p_c, 1]$. This follows easily from the estimates of the second sentence of Theorem 8.1, the polynomial growth of L (which holds because L is quasi-isometric to $\mathbb{R}^2$ [9]), and the Weierstrass M-test. $\square$

9 Continuum Percolation

In this section we will prove analyticity results for the Boolean model in $\mathbb{R}^2$ analogous to Theorem 7.1, answering a question of [33].

Let P_λ be a Poisson point process in $\mathbb{R}^d$ of intensity λ and let $\mathcal{N}(B)$ denote the number of points inside a bounded subset B of $\mathbb{R}^d$. The Boolean model is obtained by taking the union $\mathcal{Z}$ of disks of radius r, called *grains*, centred at the points of P_λ. The random radii are independent random variables and have the same distribution as another positive random variable ρ. They are also independent from P_λ. We denote (P_λ, ρ) the Boolean model with random radii sampled from ρ. If ρ is equal to a positive constant r we will write (P_λ, r).

The random set $\mathcal{Z}$ is called the *occupied region* and its complement $\mathcal{V}$ is called the *vacant region*. We will denote by $W(0)$ the connected component of $\mathcal{Z}$ containing 0 ($W(0) = \emptyset$ if 0 is not occupied) and $V(0)$ the connected component of $\mathcal{V}$ containing 0 ($V(0) = \emptyset$ if 0 is occupied).

It is well known that there is a critical value λ_c such that for every $\lambda > \lambda_c$ there is almost surely a (unique) occupied unbounded connected component Z_∞, but no unbounded connected components exist whenever $\lambda < \lambda_c$. An important tool in the study of Z_∞ is the *perolation density* $\theta_0 := \mathbb{P}_\lambda(0 \in Z_\infty)$ of Z_∞ (also called 'volume fraction' or 'percolation function'). For an introduction to the subject see [42, 45].

Under general assumptions on the grain distribution, θ_0 is continuous for every $\lambda \neq \lambda_c$ and $d \geq 2$, and $\theta_0(\lambda_c) = 0$ when $d = 2$ [42]. Similarly to the standard percolation model on $\mathbb{Z}^2$ it is expected that the latter holds for every $d \geq 3$ as well.

Much more is known about the behaviour of θ_0 on the interval (λ_c, ∞). Recently, it has been proved in [33] that θ_0 is infinitely differentiable on (λ_c, ∞) under general assumptions on the grain distribution. The authors asked whether θ_0 is analytic in that interval, and we answer this question in the affirmative when $d = 2$. For simplicity we will assume that all discs have radius 1, although our proof easily extends to the case where the radii are bounded above and below.

Theorem 9.1. *Consider the Boolean model $(P_\lambda, 1)$ in $\mathbb{R}^2$. Then θ_0 is analytic on (λ_c, ∞).*

The proof of Theorem 9.1 will follow the lines of that of Theorem 7.1. One of the main tools in the proof of the latter is the exponential decay of the probability $\mathbb{P}_p$(some $S \in \mathcal{MS}_n$ occurs), which follows from the Aizenman-Barsky property, duality, and the BK inequality. In the case of the Boolean model we will define another notion of outer-interface and our goal once again is to show that the probability of having large multi-interfaces decays exponentially in their size. However, the Boolean model lacks a notion of duality which leads to certain complications. Nevertheless, it is still true that the probability $\mathbb{P}_\lambda(\mu(V(0)) \geq a)$, where $\mu(V(0))$ denotes the area of $V(0)$, decays exponentially in a for every fixed $\lambda > \lambda_c$, which we will combine with the more general Reimer inequality [34], instead of the BK inequality, to show the desired exponential decay.

Before stating the Reimer inequality let us fix some notation. We denote a sample of the Boolean model (P_λ, ρ) by $\omega = \{(x_i, r_i) : i = 1, 2, \dots\}$, where (x_i)

is the sequence of points of the Poisson point process and (r_i) the associated sequence of radii. The *restriction* of ω to a set $K \subset \mathbb{R}^d$ is

$$\omega_K := \{(x_i, r_i) \in \omega : x_i \in K\}.$$

We also define

$$[\omega]_K := \{\omega' : \omega'_K = \omega_K\}.$$

We say that an event A *lives on* a set U if $\omega \in A$ and $\omega' \in [\omega]_U$ imply $\omega' \in A$. For A and B living on a bounded region U we define the event

$$\begin{aligned} A\Box B = \{\omega :\ & \text{there are disjoint sets } K, L,\ \text{each a finite union of} \\ & \text{rectangles with rational coordinates, with } [\omega]_K \subset A, [\omega]_L \subset B\}. \end{aligned} \tag{15}$$

When $A\Box B$ occurs we say that A and B *occur disjointly*.

Theorem 9.2. *(Reimer inequality)[34] Let U be a bounded measurable set in $\mathbb{R}^d$. For any two events A and B living on U we have*

$$\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B).$$

Before delving into the details of the proof of Theorem 9.1 let us give some more definitions. Let $x \in \mathbb{R}^2$ and let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise C^1 boundary (the sets Ω we will consider are finite unions of disks). We define $dist(x, \Omega) = \inf_{y\in\Omega}\{|x - y|\}$ to be the Hausdorff distance between x and Ω. The area of Ω is denoted by $\mu(\Omega)$ and the length of its boundary $\partial\Omega$ by $\mathcal{L}(\partial\Omega)$.

The *Minkowski sum* of two sets $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ is defined as the set

$$\Omega_1 + \Omega_2 := \{a + b : a \in \Omega_1, b \in \Omega_2\}.$$

We also define

$$r\Omega := \{ra : a \in \Omega\}$$

for $r \in \mathbb{R}_{\geq 0}$. For $x \in \mathbb{R}^2$ we will write $x + \Omega$ instead of $\{x\} + \Omega$. Analogously, the *Minkowski difference* is defined as the set

$$\Omega_1 - \Omega_2 := \{x \in \mathbb{R}^2 : x + \Omega_2 \subset \Omega_1\}.$$

Note that in general $(\Omega_1 - \Omega_2) + \Omega_2 \neq \Omega_1$. However, for the kind of sets we will consider equality will hold.

For $r \in \mathbb{R}_{\geq 0}$ the *outer r-parallel set* of Ω is the set

$$\Omega_r := \Omega + r\overline{D},$$

where $\overline{D} = \overline{D(0,1)}$ is the closed unit disk. We will write $\overline{D(x)}$ for the closed unit disk centred at x. Notice that Ω_r coincides with the set

$$\{x \in \mathbb{R}^2 : dist(x, \Omega) \leq r\}.$$

Moreover it follows by the definitions that $(\Omega_r)_s = \Omega_{r+s}$.

The *inner r-parallel set* of Ω is the set

$$\Omega_{-r} := \Omega - r\overline{D}.$$

This set could be empty for some value of r and for this reason we define the *inradius* $r(\Omega)$ of Ω by

$$r(\Omega) := \sup\{r : \exists x \in \mathbb{R}^2 \text{ with } x + r\overline{D} \subset \Omega\}.$$

Given $Y = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^2$ we define

$$\Omega(Y) := \cup_{i=1}^{n} \overline{D(x_i)}.$$

In case $\Omega(Y)$ is not simply connected, consider the bounded connected components $C_1, C_2, \ldots, C_k$ of its complement and define

$$\tilde{\Omega} = \tilde{\Omega}(Y) := (\cup_{i=1}^{k} C_k) \cup \Omega(Y).$$

The next theorem upper bounds the measure of Ω_r in terms of the measure of Ω and the length of its boundary. It will be useful in the proof of Theorem 9.1.

Theorem 9.3. ***(Steiner's inequality)*** *Let $\Omega \subset \mathbb{R}^2$ be a compact simply connected set with piecewise C^1 boundary. Then*

$$\mu(\Omega_r) \leq \mu(\Omega) + \mathcal{L}(\partial\Omega)r + \pi r^2.$$

If Ω is convex, then this inequality holds with equality.

See [24] for a proof when Ω is convex [35, 52] for the general case.

Let us now focus on the function θ_0. If $0 \notin Z_\infty$, then there are two possibilities:

(i) either there is no point of P_λ in $\overline{D}$,

(ii) or there are points $x_1, x_2, \ldots, x_n$ of P_λ in W such that $\Omega := \Omega(\{x_1, \ldots, x_n\})$ is connected and there is no point of $P_\lambda \setminus \{x_1, \ldots, x_n\}$ at distance $r \leq 1$ from $\partial\tilde{\Omega}$.

This observation leads to the following definition. Suppose that $Y = \{x_1, x_2, \ldots, x_n\}$ is a subset of $\mathbb{R}^2$ satisfying

(i) $\Omega := \Omega(Y)$ is connected;

(ii) $0 \in \tilde{\Omega}$; and

(iii) $\overline{D(x_i)} \cap J$ contains an arc of positive length for every $i = 1, \ldots, n$.

Then we call $\partial\tilde{\Omega}$ a *outer-interface* and we denote it by $J(Y)$. The set $S(Y) := J(Y) + \overline{D}$ is called a *separating strip*. We say that a set Y as above *happens to separate in* P_λ if $Y \subset P_\lambda$ and no other point of $S(Y)$ belongs to P_λ. We say that $S(Y)$ *occurs* whenever Y happens to separate in P_λ.

There is subtle point in the latter definition. It is possible that the boundary of $S(Y)$ contains points of the Jordan domain enclosed by $J(Y)$ ($\tilde{\Omega}$ with the above notation) that do not belong in Y. Moreover, it can happen that some of these points are occupied. However, having such a Y in P_λ is an event of measure 0 and so we can disregard it.

To avoid such trivialities, we will always assume that no pair of points x_i, x_j of P_λ have distance 2, which implies that no pair of disks touch. We can do so as this event has measure 0.

The following lemma is an easy consequence of the definitions.

Lemma 9.4. *If Y_1 and Y_2 happen to separate in P_λ and $S(Y_1), S(Y_2)$ have non empty intersection, then $Y_1 = Y_2$.* □

This leads us to define a *multi-interface* as a finite set of pairwise disjoint outer-interfaces and a *separating multi-strip* as a finite set of pairwise disjoint separating strips. A separating multi-strip *occurs* if each of its separating strips occurs.

Using the above definitions we obtain

$$1 - \theta_0(\lambda) = \mathbb{P}_\lambda(0 \notin Z_\infty) = \mathbb{P}_\lambda(\text{some } S(Y) \text{ occurs})$$

for every $\lambda > \lambda_c$. The second equality follows from the fact that whenever $0 \notin Z_\infty$ and no Y happens to separate in P_λ, 0 belongs to an infinite vacant component, and this event has measure 0 for every $\lambda > \lambda_c$ [42].

Once again we intend to use the inclusion-exclusion principle to obtain the formula

$$\mathbb{P}_\lambda(\text{some } S(Y) \text{ occurs}) = \sum_{k=1}^{\infty} (-1)^{k+1} \mathbb{E}_\lambda(N(k))$$

for every $\lambda \in (\lambda_c, \infty)$, where $N(k)$ is the number of occurring separating multi-strips comprising k separating strips.

To prove the validity of the above formula we will show that the alternating sum converges absolutely. In order to do so, we first express the above expectations as an infinite sum according to the area of $S(Y_i)$, i.e.

$$\mathbb{E}_\lambda(N(k)) = \sum_{\{m_1,\ldots,m_k\}} \mathbb{E}_\lambda(N(k, \{m_1, \ldots, m_k\})),$$

where the sum in the right hand side ranges over all multi-sets of positive integers with k elements, and $N(k, \{m_1, \ldots, m_k\})$ is the number of occurring separating multi-strips $S = \{S_1, \ldots, S_k\}$ with $\lfloor \mu(S_i) \rfloor = m_i$.

Let us define P_n to be the set of partitions of n and $\mathcal{MS}_n$ to be the set of separating multi-strips $S = \{S_1, \ldots, S_k\}$ with $\lfloor \mu(S_1) \rfloor + \ldots + \lfloor \mu(S_i) \rfloor = n$. We denote by N_n the number of occurring separating multi-strips of $\mathcal{MS}_n$. The analogue of Lemma 7.8 is

Lemma 9.5. *For every $\lambda \in (\lambda_c, \infty)$ there are constants $c_1 = c_1(\lambda)$ and $c_2 = c_2(\lambda)$ with $c_2 < 1$ such that for every $n \in \mathbb{N}$,*

$$\mathbb{E}_\lambda(N_n) \leq c_1 c_2^n. \tag{16}$$

Notice that whenever a separating strip S occurs, a subset of S is vacant. Thus we are lead to use the exponential decay in a of the probability $\mathbb{P}_\lambda(\mu(V(0)) \geq a)$ for every $\lambda > \lambda_c$ [42]. However, we cannot directly apply the aforementioned exponential decay as it is possible for the area of the vacant subset of S to be relatively small compared to the area of S.

In order to overcome this difficulty we fix a $\lambda > \lambda_c$ and consider a small enough $1 > \varepsilon > 0$ such that $\lambda_c(B_{1-\varepsilon}) < \lambda$, where $\lambda_c(B_{1-\varepsilon})$ is the critical point of the Poisson Boolean model $(P_\lambda, 1-\varepsilon)$. We couple the two models by sampling a Poisson point process with intensity λ in $\mathbb{R}^2$ and placing two disks, one of radius 1 and another of radius $1-\varepsilon$, centred at each point of the process. We notice that whenever a separating strip $S = S(Y)$ occurs in $(P_\lambda, 1)$, the set

$S(\varepsilon) := J(Y) + D(0, \varepsilon)$ is vacant in $(P_\lambda, 1-\varepsilon)$ in our coupling and our goal is to show that this happens with probability that decays exponentially in the area of S.

First we need to show that $\mu(S(\varepsilon))$ and $\mu(S)$ are of the same order. We do so in the following purely geometric lemma.

Lemma 9.6. *Let $1 > \varepsilon > 0$. Then there are constants $\gamma_1 = \gamma_1(\varepsilon) > 0, \gamma_2 = \gamma_2(\varepsilon) > 0$ such that for every separating strip $S = S(Y)$ we have*

$$\mu(S(\varepsilon)) \geq \gamma_1 \mu(S) - \gamma_2.$$

Proof. Let $J = J(Y)$ be the corresponding outer-interface of S. Easily, we can assume that J is not a single circle. We define $\Omega = \Omega(Y)$ to be the closure of the Jordan domain bounded by J. Let $S_{-1}(\varepsilon)$ be the intersection of $S(\varepsilon)$ with Ω. We will show that

$$\mu(\Omega_1) - \mu(\Omega) \leq 2(\mu(\Omega) - \mu(\Omega_{-1})) + \pi \tag{17}$$

and

$$\mu(S_{-1}(\varepsilon)) \geq d(\mu(\Omega) - \mu(\Omega_{-1})) \tag{18}$$

for some constant $d = d(\varepsilon) > 0$ independent of S. Then the assertion follows immediately, as $\mu(S) = \mu(\Omega_1) - \mu(\Omega_{-1})$.

For inequality (17) it suffices to prove that

$$\mathcal{L}(J) \leq 2(\mu(\Omega) - \mu(\Omega_{-1})) \tag{19}$$

because by Steiner's inequality (Theorem 9.3) we have

$$\mu(\Omega_1) \leq \mu(\Omega) + \mathcal{L}(J) + \pi.$$

For every $x \in Y$ the intersection of J with the the circle $C(x)$ of radius 1 centred at x may contain several connected components. Let (J_i) be an enumeration of all these connected components and (x_i) the corresponding sequence of centres, i.e. x_i is the centre of the arc J_i (some $x \in Y$ may appear more than once). Every arc J_i has two endpoints A_i, B_i and each endpoint $E_i \in \{A_i, B_i\}$ belongs to two disks $\overline{D(x_i)}$ and $\overline{D(x_i')}$ for some $i' = i'(E_i)$.

Let $S(i)$ be the open sector of $D(x_i)$ enclosed by the radii x_iA_i, x_iB_i and the arc J_i. Notice that $S(i)$ is a subset of $\Omega \setminus \Omega_{-1}$. We claim that any two distinct $S(i), S(j)$ are disjoint. To see this, let $x_{i'}$ be the second center that has distance 1 from E_i. Observe that no centres $x \in Y$ belong to the open disk $D(E_i)$, where $E_i \in \{A_i, B_i\}$, because otherwise E_i would not belong to the boundary of S. Moreover, every segment E_kx_j that intersects E_ix_i has to intersect $C(x_i)$ as well, because E_k belongs to the boundary of S and thus it does not belong to the open disk $D(x_i)$. Hence if E_kx_j intersects E_ix_i, then x_j is at distance at most 1 from $C(x_i)$. It is easy to deduce geometrically that for every $P \in C(x_i)$ the only points Q of $\overline{D(P)} \setminus \{x_i\}$ such that QP intersects E_ix_i belong to $D(E_i)$ (see Figure 2), which implies that the $S(i)$'s are disjoint.

These observations imply that

$$\sum_i \mu(S(i)) \leq \mu(\Omega) - \mu(\Omega_{-1}).$$

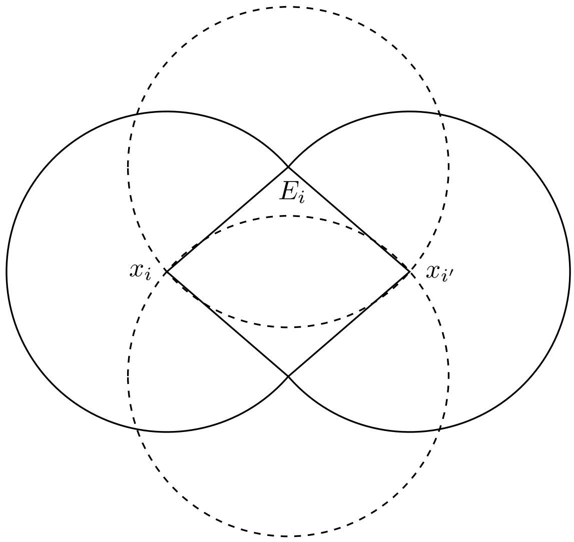

Figure 2: Four disks of radius 1 centred at $x_i, x_{i'}, E_i$ and another point of $C(x_i)$.

An elementary computation yields $\mathcal{L}(J_i) = 2\mu(S_i)$, which implies that

$$\mathcal{L}(J) = \sum_i \mathcal{L}(J_i) = 2\sum_i \mu(S(i)) \leq 2(\mu(\Omega) - \mu(\Omega_{-1}))$$

establishing (19).

For inequality (18) we will assume for technical reasons that $\varepsilon < 1/2$. The case $\varepsilon \geq 1/2$ follows readily, because $S_{-1}(\varepsilon)$ increases as ε increases.

We will split both S_{-1} and $S_{-1}(\varepsilon)$ into several smaller pieces. Let us first focus on S_{-1}. The two radii $E_i x_i$ and $E_i x_i'$ that emanate from the endpoint $E_i \in \{A_i, B_i\}$ of J_i define an open sector $T(E_i)$ of $D(E_i)$. By the definitions, the collection of all the $\overline{T(E_i)}$'s together with all the $\overline{S(i)}$'s cover S_{-1} (see Figure 3). The elements of the collection are not necessarily pairwise disjoint, but this works only in our favour as we need a mere upper bound for the area of S_{-1}.

We will now compare the areas of $S(i)$ and $T(E_i)$ with those of their subsets $S(i,\varepsilon) = S(i) \cap S_{-1}(\varepsilon)$ and $T(E_i,\varepsilon) = T(E_i) \cap S_{-1}(\varepsilon)$. As the sectors $S(i)$ do not intersect, the sets $S(i,\varepsilon)$ do not intersect either. It is a matter of simple calculations to see that

$$\mu(S(i,\varepsilon)) = (1 - (1-\varepsilon)^2)\mu(S(i)). \tag{20}$$

On the other hand, the $T(E_i,\varepsilon)$'s may intersect. Our goal is to associate to every $T(E_i)$ a domain $\Omega(E_i)$ that contains $T(E_i)$ and every other $T(E_j)$ such that $T(E_j,\varepsilon)$ intersects $T(E_i,\varepsilon)$. Later on we will be generous and keep only some $\Omega(E_i)$ that we need to cover S_{-1}. In order to define $\Omega(E_i)$, notice first that whenever $T(E_i,\varepsilon)$ and $T(E_j,\varepsilon)$ intersect, E_j has distance at most $2\varepsilon < 1$ from E_i. Hence any other point of $T(E_j)$ has distance at most $1 + 2\varepsilon$ from E_i. Consider the points $y = y(E_i, x_i, \varepsilon)$ and $y' = y'(E_i, x_{i'}, \varepsilon)$ in $E_i x_i$ and $E_i x_{i'}$, respectively, that have distance 2ε from E_i (see Figure 4). Extend each

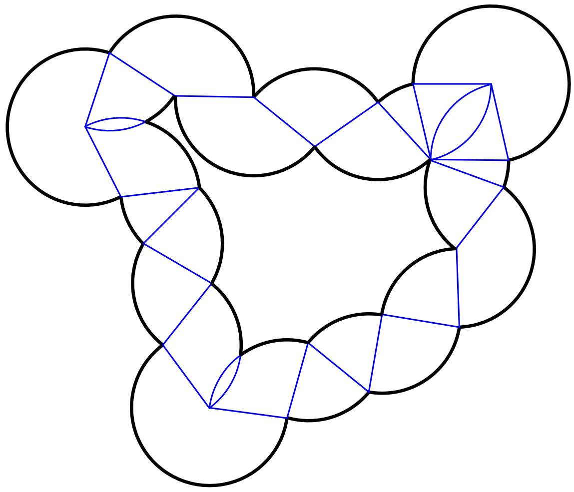

Figure 3: The domain S_{-1} enclosed by the the black curves and the sectors $T(E_i)$ enclosed by the blue radii and the blue/black arcs.

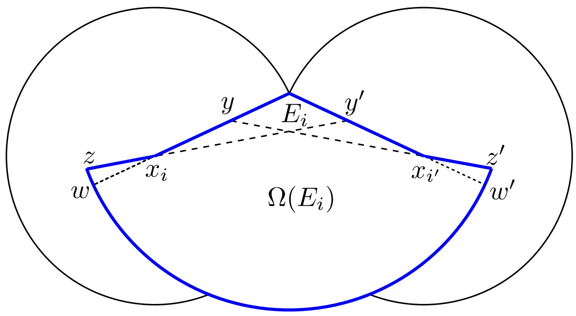

Figure 4: The domain $\Omega(E_i)$.

of $yx_{i'}, y'x_i, E_ix_i$ and $E_ix_{i'}$ up to distance $1+2\varepsilon$ from E_i, and let z', z, w and w' be the endpoints of these new segments. Define $\Omega(E_i)$ as the domain enclosed by the segments $E_ix_i, E_ix_{i'}, x_iz, x_{i'}z'$ and the arc of the circle $C(E_i, 1+2\varepsilon)$ from z to z' that contains w and w'.

It is easy to see from the construction of $\Omega(E_i)$ that any $T(E_j)$ such that $T(E_j,\varepsilon)$ intersects $T(E_j,\varepsilon)$, is contained in Ω_i. This follows from the fact that the $S(i)$'s are disjoint as proved above, and so no other sector $T(E_j)$ intersects E_ix_i or $E_ix_{i'}$.

We claim that there is a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\mu(T(E_i,\varepsilon)) \geq \delta\mu(\Omega(E_i)) \tag{21}$$

for every i. Indeed, the area of the sector $S(E_i, w, w')$ of $D(E_i, 1+2\varepsilon)$ bounded by the radii E_iw and E_iw' is of the same order as the area of $\Omega(E_i)$, because the angles of the segments x_iw, x_iz and $x_{i'}w', x_{i'}z'$ are of the same order as the angle θ of the segments $E_ix_i, E_ix_{i'}$. Moreover, there is some constant $\Theta = \Theta(\varepsilon) > 0$ such that if θ is smaller than Θ, then x_i and $x_{i'}$ are close enough that the sector of $D(E_i,\varepsilon)$ defined by the segments E_ix_i and $E_ix_{i'}$ is contained in $S_{-1}(\varepsilon)$. A simple computation shows that the area of this sector is of the same order as the area of $S(E_i, w, w')$. On the other hand, $\mu(T(E_i, \varepsilon))$ is bounded from below by a strictly positive constant for every $\theta \geq \Theta$. Combining all the above we conclude that (21) holds.

Let us consider a set F of endpoints that is maximal with respect to the property that $T(E_i, \varepsilon)$ and $T(E_j, \varepsilon)$ do not intersect for any $E_i, E_j \in F$ with $i \neq j$. The maximality of F implies that the collection $\mathcal{S}$ of all the $S(i)$'s together with the collection $\mathcal{O}$ of the $\Omega(E_i)$'s for $E_i \in F$ cover S_{-1}, because for any other set $T(E_k)$ with $E_k \notin F$, $T(E_k, \varepsilon)$ intersects some $T(E_i, \varepsilon)$ with $E_i \in F$ and thus $T(E_k)$ is contained in $\Omega(E_i)$. However, it is possible that some element of $\mathcal{S}$ intersects some element of $\mathcal{O}$. Nevertheless, each intersection point is counted exactly twice, because the elements of $\mathcal{S}$ and $\mathcal{O}$ are disjoint. Hence

$$\mu(S_{-1}(\varepsilon)) \geq 1/2\big(\sum_i \mu(S(i, \varepsilon)) + \sum_{x \in F} \Omega(x)\big),$$

which combined with (20) and (21) implies inequality (18). □

Notice that every $S(\varepsilon)$ has a non-empty intersection with the non-negative real line $[0, \infty)$, because S has this property. In fact if x is the point of $J \cap [0, \infty)$ which has greatest distance from 0, where J is the outer-interface that defines S, then the interval $[x, x + \varepsilon)$ is contained in $S(\varepsilon) \cap [0, \infty)$. We conclude that $S(\varepsilon)$ contains one of the points $\{0, \varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$ for some $N \in \mathbb{N}$ depending on $S(\varepsilon)$. The next lemma provides a uniform upper bound for N that depends only on the area of S.

Lemma 9.7. *For every separating strip $S = S(Y)$ we have*

$$S \subset D(0, 3\mu(S)).$$

Proof. Let $J = J(Y)$ be the outer-interface that defines S, and $\Omega = \Omega(Y)$ the closure of the Jordan domain bounded by J. By the definition of J we have $0 \in \Omega$. Thus the distance of any point of J from 0 is bounded from above by $\mathcal{L}(J)$. This implies that the distance of any point in S from 0 is bounded from above by $\mathcal{L}(J) + 1$. Combining (19) with the fact that $\mu(\Omega) - \mu(\Omega_{-1}) \leq \mu(S)$ we obtain

$$\mathcal{L}(J) \leq 2\mu(S).$$

Moreover, $\mu(S) > 1$ because by definition S contains a disk of radius 1. Therefore

$$\mathcal{L}(J) + 1 < 3\mu(S).$$

Combining these inequalities yields the desired assertion. □

We deduce from Lemma 9.7 that N can be chosen to be $\lfloor 3\mu(S)/\varepsilon \rfloor$. We are now almost ready to prove the desired exponential decay. Before we do so we need to upper bound the number of occurring separating multi-strips of $\mathcal{MS}_n$.

Lemma 9.8. *There is a constant $R \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ at most $R^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any ω.*

Proof. Notice that a separating strip $S = S(Y)$ contains an interval of the form $[x, x + 1]$ for some $x \in [0, \infty)$. Combined with Lemma 9.7 this implies that S contains some element of the set $\{0, 1, \ldots, \lfloor 3\mu(S) \rfloor\}$. We can now proceed as in the proof of Lemma 7.7. □

We are now ready to prove Lemma 9.5.

Proof of Lemma 9.5. Since

$$N_n \le R^{\sqrt{n}} \chi_{\{\text{some } S \in \mathcal{MS}_n \text{ occurs}\}}$$

by Lemma 9.8, we conclude that

$$\mathbb{E}_\lambda(N_n) \le R^{\sqrt{n}} \mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs}).$$

Hence it suffices to show that $\mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs})$ decays exponentially.

Recall our coupling between the Boolean models $(P_\lambda, 1)$ and $(P_\lambda, 1-\varepsilon)$, and the fact that whenever Y happens to separate in P_λ the set $S(\varepsilon)$ is a vacant connected subset of $(P_\lambda, 1-\varepsilon)$ in our coupling. For $m \in \mathbb{N}$, let $V(m)$ denote the event that there is a subset V of a vacant component with $\mu(V) \ge \gamma_1 m - \gamma_2$, where γ_1, γ_2 are the constants of Lemma 9.6, and some element of the set $\{0, \varepsilon, \dots, \lfloor (3m+3)/\varepsilon \rfloor \varepsilon\}$ belongs to V, and V is contained in $D(0, 3m+3)$. We claim that

$$\mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs}) \le \sum_{\{m_1, m_2, \dots, m_k\} \in P_n'} \mathbb{P}_{\lambda, 1-\varepsilon}(V(m_1) \square \dots \square V(m_k)),$$

where as above $\square$ means that the events occur disjointly, P_n' is the set of partitions of n with the property that for every $N \le n$ at most $3N+3$ elements of the partition have size at most N, and the probability measure $\mathbb{P}_{\lambda,1-\varepsilon}$ refers to the Boolean model $(P_\lambda, 1-\varepsilon)$. The upper bound $3N+3$ on the number of elements of size at most N comes from the fact that any separating strip $S = S(Y)$ contains some element of the set $\{0, 1, \dots, \lfloor 3\mu(S) \rfloor\}$, as remarked in the proof of Lemma 9.8. The inequality follows similarly to (13).

Reimer's inequality [34] states that

$$\mathbb{P}_{\lambda,1-\varepsilon}(V(m_1) \square \dots \square V(m_k)) \le \mathbb{P}_{\lambda,1-\varepsilon}(V(m_1)) \cdot \ldots \cdot \mathbb{P}_{\lambda,1-\varepsilon}(V(m_k)).$$

Combining the fact that $\mathbb{P}_{\lambda,1-\varepsilon}(\mu(V(0)) \ge a) \le c^a$ [42] for every $\lambda > \lambda_c$ and some $c = c(\lambda) < 1$ with the union bound we obtain

$$\mathbb{P}_{\lambda,1-\varepsilon}(\mu(V(m)) \le c_1 {c_2}^m,$$

where $c_1 = (\lfloor (3m+3)/\varepsilon \rfloor + 1) c^{-\gamma_2}$ and $c_2 = c^{\gamma_1} < 1$. We can now argue as in the proof of Lemma 7.8 to obtain the desired exponential decay. $\square$

We proceed by establishing the analyticity and the necessary estimates of the functions involved in Lemma 9.5 that we will combine with their exponential decay to prove the analyticity of θ_0.

Given a partition $\{m_1, m_2, \dots, m_k\}$ of a number n, we define $N(\{m_1, \dots, m_k\})$ to be the number of occurring separating multi-strips $S = \{S_1, \dots, S_k\}$ such that $\lfloor \mu(S_i) \rfloor = m_i$.

Lemma 9.9. *Let $\{m_1, m_2, \dots, m_k\}$ be a partition of n. Then the function $f(\lambda) := \mathbb{E}_\lambda(N(\{m_1, \dots, m_k\}))$ admits an entire extension satisfying*

$$|f(z)| \le e^{4nM} f(\lambda + M) \tag{22}$$

for every $\lambda \ge 0$, $M > 0$ and $z \in D(\lambda, M)$.

Proof. To ease notation we will prove the assertion for $i = 2$. The general case can be handled similarly.

Given two disjoint sets $Y_1 = \{x_1, \ldots, x_{j_1}\}$ and $Y_2 = \{x_{j_1+1}, \ldots, x_{j_1+j_2}\}$, we let $L(x_1, \ldots, x_{j_1+j_2})$ denote the indicator function of the event that Y_1 and Y_2 satisfy conditions (9) and $\lfloor \mu(S_i) \rfloor = m_i$. The characteristic function of the event $\{Y_i$ happens to separate in $P_\lambda\}$ is denoted by χ_{Y_i}. Let us also define the functions

$$g(x_1, \ldots, x_{j_1+j_2}) := \mu(S(x_1, \ldots, x_{j_1})) + \mu(S(x_{j_1+1}, \ldots, x_{j_2}))$$

and

$$h(x_1, \ldots, x_{j_1+j_2}) := L(x_1, \ldots, x_{j_1+j_2}) e^{-\lambda g(x_1, \ldots, x_{j_1+j_2})}.$$

First we will find a suitable formula for f. We claim that

$$f(\lambda) = \textstyle\sum_{j_1, j_2 = 1}^{\infty} \sum_{m = j_1 + j_2}^{\infty} \dfrac{(\lambda \mu(6nD))^m}{m!} \binom{m}{j_1} \binom{m - j_1}{j_2} f(\lambda, j_1, j_2), \tag{23}$$

where

$$f(\lambda, j_1, j_2) = \textstyle\int_{6nD} \dfrac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \dfrac{dx_{j_1+j_2}}{\mu(6nD)} h(x_1, \ldots, x_{j_1+j_2}). \tag{24}$$

Indeed, notice that $N(\{m_1, m_2\}) = \sum_{Y_1 \in A_{m_1}, Y_2 \in A_{m_2}} \chi_{Y_1} \chi_{Y_2}$, where A_N is the set of those subsets Y of P_λ that satisfy conditions (9) and $\lfloor \mu(S(Y)) \rfloor = N$. Moreover, we have

$$\mu(S(Y_1)) + \mu(S(Y_2)) \le (k_1 + 1) + (k_2 + 1) \le 2k_1 + 2k_2 = 2n, \tag{25}$$

since $1 \le k_1, k_2$, which combined with Lemma 9.7, implies that $N(\{m_1, m_2\})$ depends only on the points of the Poisson point process inside the disk $6nD$. Now regard $P_\lambda \cap 6nD$ as a finite Poisson process whose total number of points has a Poisson distribution with parameter $\lambda \mu(6nD)$, each point being uniformly distributed over $6nD$. Notice that conditioned on the number of points $\mathcal{N}(6nD)$ inside $6nD$, the distribution of the sets Y_1, Y_2 depends only on their sizes.

Conditionally on the sets $Y_1 = \{x_1, \ldots, x_{j_1}\}$ and $Y_2 = \{x_{j_1+1}, \ldots, x_{j_1+j_2}\}$ being contained in P_λ, the expectation of $\chi_{Y_1} \chi_{Y_2}$ is equal to $h(x_1, \ldots, x_{j_1+j_2})$. Hence expressing f according to the number of points of the Poisson process inside $6nD$ and the size of the sets Y_1, Y_2 we obtain (23). The factors $\dfrac{(\lambda \mu(6nD))^m}{m!}$ and $\binom{m}{j_1} \binom{m - j_1}{j_2}$ correspond to $\mathbb{P}_\lambda(\mathcal{N}(6nD) = m)$ and the number of ways to choose two disjoint subsets of size j_1 and j_2 from a set of size m, respectively.

Using (23) we see that f extends to an entire function. Indeed, the assertion will follow from the standard tools once we have shown that every summand of f is an entire function and that the upper bound (22) holds for the summands of f in place of f.

First we express $e^{-\lambda g(x_1, \ldots, x_{j_1+j_2})}$ via its Taylor expansion

$$e^{-\lambda g(x_1, \ldots, x_{j_1+j_2})} = \sum_{s=0}^{\infty} \frac{(-\lambda g(x_1, \ldots, x_{j_1+j_2}))^s}{s!}.$$

We will plug this into (24). We notice that the coefficient

$$\int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} L(x_1, \ldots, x_{j_1+j_2}) (-g(x_1, \ldots, x_{j_1+j_2}))^s / s!$$

is absolutely bounded by $(2n)^s/s!$, as $g(x_1,\ldots,x_{j_1+j_2}) = \mu(S(Y_1))+\mu(S(Y_2)) \leq 2n$ by (25) and $0 \leq L(x_1,\ldots,x_{j_1+j_2}) \leq 1$. Therefore the function defined by the Taylor expansion

$$\sum_{s=0}^{\infty} \lambda^s \int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} L(x_1,\ldots,x_{j_1+j_2})(-g(x_1,\ldots,x_{j_1+j_2}))^s/s!$$

is entire and by reversing the order of summation and integration we conclude that it coincides with $f(\lambda, j_1, j_2)$.

Now let $\lambda \geq 0$ and $M > 0$. Since $|z|^m \leq (\lambda+M)^m$ for every $z \in D(\lambda, M)$, inequality (22) will follow once we prove that

$$|f(z,j_1,j_2)| \leq e^{4nM} f(\lambda+M, j_1, j_2) \text{ for every } z \in D(\lambda, M). \tag{26}$$

Using once again (25) we obtain

$$|e^{-zg(x_1,\ldots,x_{j_1+j_2})}| \leq e^{-(\lambda-M)g(x_1,\ldots,x_{j_1+j_2})} =$$
$$e^{2Mg(x_1,\ldots,x_{j_1+j_2})} e^{-(\lambda+M)g(x_1,\ldots,x_{j_1+j_2})} \leq e^{4nM} e^{-(\lambda+M)g(x_1,\ldots,x_{j_1+j_2})}.$$

Hence (26) follows from the triangle inequality. This proves (22).

Combining (26) with (23) and the theorems of Weierstrass in the Appendix imply that f is analytic as well. $\square$

We are finally ready to prove Theorem 9.1.

Proof of Theorem 9.1. Consider the functions

$$f(\lambda) = \sum_{k=1}^{\infty} (-1)^{k+1} \mathbb{E}_\lambda(N(k))$$

and

$$g_n(\lambda) := \sum_{\{m_1,m_2,\ldots,m_k\} \in P_n} (-1)^{k+1} \mathbb{E}_\lambda(N(\{m_1,\ldots,m_k\}).$$

Notice that

$$f = \sum_{n=1}^{\infty} g_n.$$

By Lemma 9.5 we have

$$\sum_{k=1}^{\infty} \mathbb{E}_\lambda(N(k)) < \infty$$

for any $\lambda > \lambda_c$. Hence f coincides with $1-\theta_0$ on the interval (λ_c, ∞) by the inclusion-exclusion principle as remarked above. Combining Lemma 9.5 with Lemma 9.9 we conclude that for every $\lambda > \lambda_c$ there are constants $M = M(\lambda) > 0$, $c_1 = c_1(\lambda) > 0$ and $0 < c_2 = c_2(\lambda) < 1$ such that $|g_n(z)| \leq c_1 {c_2}^n$ for every $z \in D(\lambda, M)$. As usual, by the theorems of Weierstrass in the Appendix we conclude that f, and thus θ_0, is analytic on the interval (λ_c, ∞). $\square$

10 Finitely presented groups

In this section we will prove that $p_{\mathbb{C}} < 1$ holds for every finitely presented Cayley graph. The ideas used involve a refinement of Peierls' argument as in Timar's proof [51] of the theorem of Babson & Benjamini [10] that $p_c < 1$ for those graphs, combined with the ideas of Section 7. We start with a sketch of these ideas.

Peierls' classical argument for proving e.g. that $p_c < 1$ for bond percolation on a planar lattice G goes as follows. If the cluster $C(o)$ of the origin o is finite in a percolation instance, then $C(o)$ is surrounded by a 'cut' of vacant edges, which form a cycle in the dual lattice G^*. But the number of candidate cycles of G^* with length n is at most d_*^n, where d_* is the degree of G^*, and each of them occurs with probability $(1-p)^n$ in a percolation instance. Therefore, the union bound implies that we can make the probability that at least one of them occurs smaller than 1 if we choose p is close enough to 1, because the exponential decay of $(1-p)^n$ outperforms the at most exponential growth of the number of candidate cycles.

For this argument it was not crucial that the cut separating $C(o)$ from infinity was a cycle: to deduce that there are at most c^n candidate cuts for some constant c, it suffices if the edges of any such cut B are close to each other in the following sense. If we build an auxiliary graph, with vertex set B, by connecting any two edges of B with an edge whenever their distance is at most some bound, then this auxiliary graph is connected. For if this is the case, then using the fact that every regular graph has at most exponentially many connected subgraphs containing a fixed vertex and n further vertices (see Section 14), we deduce that there are at most c^n candidates for our B. This is the aforementioned argument of Timar [51]. The upper bound on the closeness of the edges of B arises from the length of the longest relator in the group-presentation of G.

Since Peierls' argument relies on the union bound, and many candidate cuts can occur simultaneously in a percolation instance, it is not good enough for our purposes because we need equalities rather than inequalities in formulas like (11), where we add probabilities of events similar to the event that a cut as above occurs. To prove that $p_{\mathbb{C}} = p_C$ in the planar case we therefore considered ... rather than the cut separating $C(o)$ from infinity. A ... consists of a connected (occupied) subgraph I_O of $C(o)$, namely the boundary of its unbounded face, as well as the set I_V of (vacant) edges disconnecting I_O from infinity.

Most of the work of this section is devoted to combining these two ideas in the setup of a finitely presented Cayley graph G. We introduce a notion of interface (I_V, I_O) with the following properties. Every finite percolation cluster C of G is 'bounded' by such an interface (I_V, I_O), where I_V consists of the vacant edges separating C from infinity, and I_O defines a connected sub-cluster of C, incident with all edges in I_V. So far this is trivial to satisfy, as we could have taken $I_O = C$. But we need the size of I_V to be proportional to that of I_O in order to use a Peierls-type argument, so we need I_O to be a 'thin' layer near the boundary I_V of C. In addition, we need (I_V, I_O) to be uniquely determined by C in order to express θ in an equality like (11) (see (30) below). Moreover, we need the event that (I_V, I_O) is an interface of some cluster in a percolation instance to depend on the state of the edges in $I_V \cup I_O$ only, in order to have a formula (of the form $p^{|I_O|}(1-p)^{|I_V|}$) for the probability of this event that we can do our complex analysis with. (Some complications here are imposed by the fact that we will use an inclusion-exclusion formula as above.) Finally, we need $I_V \cup I_O$ to span a connected subgraph of G, in order to guarantee that there are at most exponentially many 'candidate' interfaces of $C(o)$, as in Timar's aforementioned proof.

Satisfying all these properties at once is non-trivial, as we need the balance of choices between too large and too small subgraphs of $C \cup \partial C$ to stabilise at a uniquely determined middle. After some preliminaries, we offer our notion of interface in Definition 10.3, followed by proofs of the aforementioned properties. We then exploit our notion to prove our analyticity results in Section 10.5.

The reader wishing to get a feeling of the results of this section without all their combinatorial details may do so by reading Section 10.2 up to Definition 10.1, Section 10.3, the statement of Proposition 10.4, perhaps the proof of Proposition 10.5, and as much of Section 10.5 needed to be convinced that the above proof ideas can be carried out along the lines of the proof of the planar case.

10.1 The setup and notation

The *edge space* of a graph G is the direct sum $\mathcal{E}(G) := \bigoplus_{e \in E(G)} \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{0,1\}$ is the field of two elements, which we consider as a vector space over $\mathbb{Z}_2$. (By coincidence, $\mathcal{E}(G)$ is also our space of percolation configurations.) The cycle space $\mathcal{C}(G)$ of G is the subspace of $\mathcal{E}(G)$ spanned by the *circuits* of cycles, where a circuit is an element $C \in \mathcal{E}(G)$ whose non-zero coordinates $\{e \in E(G) \mid C_e = 1\}$ coincide with the edge-set of a cycle of G.

Let $P = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a group presentation, and let $G = Cay(P)$ be the corresponding Cayley graph. Let $\mathcal{P}$ be the set of closed walks of G induced by the relators in $\mathcal{R}$. It is straightforward to prove that $\mathcal{P}$ forms a basis of the cycle space $\mathcal{C}(G)$ of G.

More generally, we can let G be an arbitrary graph, and let $\mathcal{P}$ be any basis of $\mathcal{C}(G)$. For the applications of the theory developed in this section to percolation it will be important for G to be of bounded degree and 1-ended, and for the elements of $\mathcal{P}$ to have a uniform upper bound on their size.

We will assume for simplicity that all elements of $\mathcal{P}$ are cycles (rather than more general closed walks with self-intersections); this assumption comes without loss of generality.

We let $vw = wv$ denote the edge of G joining two vertices v and w. Every edge $e = vw \in E(G)$ has two *directions* $\vec{vw}, \vec{wv}$, which are the two directed sets comprising v, w. The head $head(\vec{vw})$ of $\vec{vw}$ is w.

For $F \subset E(G)$ we let $\overleftrightarrow{F}$ denote the set of directions of the edges of F. Thus $|\overleftrightarrow{F}| = 2|F|$. In particular, $\overleftrightarrow{E}(G)$ denotes the set of directed edges of G.

A percolation instance is an element ω of $\Omega = \{0,1\}^{E(G)}$.

10.2 A connectedness concept

We say that (B_1, B_2) is a *proper bipartition* of a set B, if $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$ and $B_1, B_2 \neq \emptyset$.

Recall that Timar's argument involved the idea that the edges of the cut B separating $C(o)$ from infinity form a connected auxiliary graph. This can be reformulated by saying that for every proper bipartition (B_1, B_2) of B, there are edges $b_1 \in B_1, b_2 \in B_2$ that are 'close' to each other. The measure of closeness used was that there is a relator in the presentation inducing a cycle containing both (in particular, b_1, b_2 are then close in graph distance). We use a similar idea here, but for technical reasons we need to reformulate this in the language of directed edges.

A *$\mathcal{P}$-path connecting* two directed edges $\vec{vw}, \vec{yx} \in \overset{\leftrightarrow}{E}(G)$ is a path P of G such that the extension $vwPyx$ is a subpath of an element of $\mathcal{P}$. Here, the notation $vwPyx$ denotes the path with edge set $E(P) \cup \{vw, yx\}$, with the understanding that the endvertices of P are w, y. Note that P is not endowed with any notion of direction, but the directions of the edges $\vec{vw}, \vec{yx}$ it connects do matter. We allow P to consist of a single vertex $w = y$.

We will say that P *connects* an undirected edge $e \in E(G)$ to $\vec{f} \in \overset{\leftrightarrow}{E}(G)$ (respectively, to a set $J \subset \overset{\leftrightarrow}{E}(G)$), if P is a $\mathcal{P}$-path connecting one of the two directions of e to $\vec{f}$ (resp. to some element of J).

Definition 10.1. *We say that a set $J \subset \overset{\leftrightarrow}{E}(G)$ is F-connected for some $F \subset E(G)$, if for every proper bipartition (J_1, J_2) of J, there is a $\mathcal{P}$-path in $G - F$ connecting an element of J_1 to an element of J_2.*

As usual, a notion of 'connectedness' gives rise to a corresponding notion of 'components'. In our case, an *F-component* of any set $K \subset \overset{\leftrightarrow}{E}(G)$ is a maximal F-connected subset of K. It is an immediate consequence of the definitions that if two sets $J, J' \subset \overset{\leftrightarrow}{E}(G)$ are both F-connected, and their intersection is non-empty, then $J \cup J'$ is F-connected too. Therefore,

the F-components of K form a partition of any $K \subset \overset{\leftrightarrow}{E}(G)$. (27)

This implies the following monotonicity property of F-components.

Proposition 10.2. *If $Y \subset \overset{\leftrightarrow}{E}(G)$ is contained in an F-component of some $J \subset \overset{\leftrightarrow}{E}(G)$ (with $J \supseteq Y$), then Y is contained in an F'-component of J' whenever $F' \subseteq F$ and $J' \supseteq J$.*

Proof. If Y is not contained in an F'-component of J', then in particular J' is not F'-connected. As J' is partitioned by its F'-components by (27), we can then find a proper bipartition (J'_1, J'_2) of J' such that both $Y \cap J'_1$ and $Y \cap J'_2$ are non-empty and there is no $\mathcal{P}$-path in $G - F'$ connecting J'_1 to J'_2. Consider then the bipartition $(J'_1 \cap J, J'_2 \cap J)$ of J, which is proper since both sides meet Y. As Y is contained in an F-component of J, there is a $\mathcal{P}$-path P in $G - F$ connecting $J'_1 \cap J$ to $J'_2 \cap J$. But $P \subset G - F'$ since $F' \subseteq F$, and it connects J'_1 to J'_2, contradicting our assumption. □

It is easy to see that

if J is F-connected, then there is a component of $G - F$ containing the head of every element of J. (28)

10.3 $\mathcal{P}$-Interfaces

Given $F \subset E(G)$ and a subgraph D of G, let $\vec{F^D} := \{\vec{vz} \mid vz \in F, z \in V(D)\}$. Thus if $f \in F \cap \partial D$ then $\vec{F^D}$ contains the direction of f towards D only, if $f \in F \cap E(D)$ then $\vec{F^D}$ contains both directions of f, and otherwise $\vec{F^D}$ contains no direction of f. Fix a vertex $o \in V(G)$.

We now give the crucial definition of this section, following the intuition sketched in the beginning of this section.

Definition 10.3. *A* $\mathcal{P}$-interface *is a pair $I = (I_V, I_O)$ of sets of edges of G with the following properties*

(i) I_V separates o from infinity;

(ii) There is a unique finite component D of $G - I_V$ containing a vertex of each edge in I_V;

(iii) $\vec{I_V^D}$ is I_V-connected; and
(Note that by (ii), $\vec{I_V^D}$ contains at least one of the two directions of each edge in I_V. It may contain both directions of some edges.)

(iv) $I_O = \{e \in E(D) \mid$ there is a $\mathcal{P}$-path in $G - I_V$ connecting e to $\vec{I_V^D}$ $\}$.
(This is equivalent to
$I_O = \{vz \in E(D) \mid \{\vec{vz}\} \cup \vec{I_V^D}$ or $\{\vec{zv}\} \cup \vec{I_V^D}$ is I_V-connected$\}$.)

Note that I_V is always non-empty, but I_O is empty when I_V consists of the set of edges incident with o. It is not hard to see that $I_O \neq \emptyset$ for all other I_V when G is 1-ended.

Clearly, I_O is uniquely determined by I_V via (iv), so any I_V satisfying the other three properties introduces a $\mathcal{P}$-interface by defining I_O via (iv). The reason why we do not define I_V alone to be the $\mathcal{P}$-interface is to satisfy the uniqueness property in Proposition 10.4 below. It follows from this definition that I_V also separates I_O from infinity.

Examples: if $\mathcal{P}$ is the standard presentation $\langle x, y \mid xy = yx\rangle$ of $\mathbb{Z}^2$, then the $\mathcal{P}$-interfaces coincide with the outer-interfaces from Section 7.

An important aspect of the definition of a $\mathcal{P}$-interface is that (vacant) edges with both endvertices in the same cluster need to be accepted in I_V to satisfy Proposition 10.9. This is why in (i) I_V is declared to be a superset of a o–∞ cut B, rather than B itself. It is a good exercise to try to visualise a $\mathcal{P}$-interface of the standard presentation of $\mathbb{Z}^3$, i.e. the cubic lattice in $\mathbb{R}^3$ presented by its 4-cycles. A further good exercise is to try to visualise how $\mathcal{P}$-interfaces of $\mathbb{Z}^2$ or $\mathbb{Z}^3$ grow as we allow further (redundant) relators in our presentation, e.g. all cycles up to a given length.

10.4 Properties of $\mathcal{P}$-interfaces

We now prove that the notion of $\mathcal{P}$-interface we introduced satisfies the many properties needed in order to carry out the Peierls-type argument sketched at the beginning of this section.

From now on we assume that

> G is an infinite, 1-ended, finitely presented Cayley graph fixed throughout, or more generally, an 1-ended bounded degree graph, admitting a basis $\mathcal{P}$ of $\mathcal{C}(G)$ whose elements are cycles of bounded lengths (as discussed in Section 10.1). (29)

We say that a $\mathcal{P}$-interface $I = (I_V, I_O)$ *occurs* in a percolation instance $\omega \in \{0,1\}^{E(G)}$, if every edge in I_O is occupied and every edge in I_V is vacant in ω.

We say that I *meets* a cluster C of ω, if either $I_O \cap E(C) \neq \emptyset$, or $I_O = E(C) = \emptyset$ and $I_V = \partial C$ (in which case C consists of o only).

Theorem 10.4. *For every finite percolation cluster C of G such that ∂C separates o from infinity, there is a unique $\mathcal{P}$-interface (I_V, I_O) that meets C and occurs. Moreover, we have $I_O \subseteq E(C)$ and $I_V \subseteq \partial C$ for that $\mathcal{P}$-interface.*

Conversely, every occurring $\mathcal{P}$-interface meets a unique percolation cluster C, and ∂C separates o from infinity (in particular, C is finite).

The proof of this is rather involved, and needs some intermediate steps which we gather now.

The following proposition is based on Timar's [51] aforementioned proof of the theorem of Babson & Benjamini [10], and contains the quintessence of the notion of a $\mathcal{P}$-interface.

A *minimal cut* of G is a minimal set of edges that disconnects G. Note that if B is a minimal cut, then $G - B$ has exactly two components, and every edge in B has an end-vertex in each of these components.

Proposition 10.5. *Let B be a minimal cut of G and let $L \subset E(G)$ be a superset of B such that some component D of $G - L$ contains a vertex of each edge in B. Then $\vec{B^D}$ is contained in an L-component of $\vec{L^D}$.*

Proof. Suppose to the contrary that there are directed edges $e, f \in \vec{B^D}$ that lie in distinct L-components of $\vec{L^D}$. Note that e, f cannot be the two directions of the same undirected edge because no edge of B has both end-vertices in D by the above remark about minimal cuts. Let (L_1, L_2) be a proper bipartition of $\vec{L^D}$ such that $e \in L_1, f \in L_2$, and there is no $\mathcal{P}$-path in $G - L$ connecting L_1 to L_2, which exists by the definitions and the fact that $\vec{L^D}$ is partitioned by its L-components by (27).

Let R be an e-f path in D, which exists because D is assumed to contain a vertex of each edge in B. Let Q be an e-f path in the component of $G - B$ avoiding D; this component exists because $G - B$ has exactly two components, one of which contains D since $L \supseteq B$ (Figure 5).

Let K be the cycle obtained by joining these paths R, Q using e and f. Since $\mathcal{P}$ is a basis for the cycle space $\mathcal{C}(G)$, we can express K as a sum $\sum C_i$ of cycles $C_i \in \mathcal{P}$, where this sum is understood as taking place in $\mathcal{C}(G)$.

Note that no cycle C_i contains a path in $G - L$ connecting L_1 to L_2, because no such path exists by the choice of (L_1, L_2). Let $L_{C_i} := \overleftrightarrow{L \cap E(C)}$ be the directions of edges of L appearing in C_i. The previous remark implies that L_{C_i} has an even number of its elements in each of L_1, L_2, because each component of $C_i - L$ (which is a subpath of C_i) is incident with either 0 or 2 such elements pointing towards the component, and they lie both in L_1 or both in L_2 or both in none of the two.

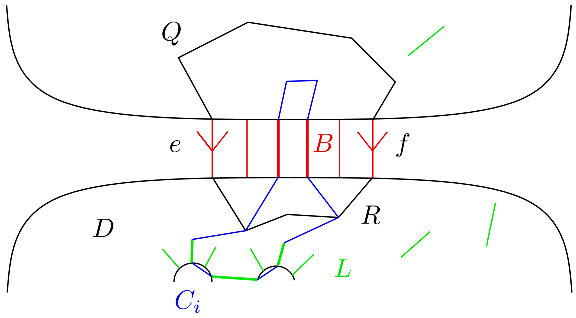

Figure 5: The situation in the proof of Proposition 10.5.

This leads into a contradiction by a parity argument: notice that our cycle K contains an odd number of directions of edges in each of L_1, L_2, namely exactly one in each —e and f respectively— because P avoids L and Q avoids D, hence $\vec{L^D}$, by definition. But then our equality $K = \sum C_i$ is impossible by the above claim because sums in $\mathcal{C}(G)$ preserve the parity of the number of (directed) edges in any set. This contradiction proves our statement. $\square$

We can use the same ideas to prove the following proposition.

Proposition 10.6. *Let $L \subseteq E(G)$, let D be a component of $G - L$, and let $e = vz$ be an edge of L such that $v, z \in V(D)$. Then $\vec{vz}, \vec{zv}$ lie in the same L-component of $\vec{L^D}$.*

Proof. It is not hard to adapt the proof of Proposition 10.5 to our setup to prove our statement; the only difference is that instead of the cycle K we now consider a cycle consisting of the edge vz and a v–z path in D. But we can in fact just apply Proposition 10.5 to an auxiliary graph to deduce Proposition 10.6 as follows. Subdivide the edge vz into two edges vw, wz by adding a new vertex w. Consider the minimal cut B of the resulting graph that consists of these two edges vw, wz (and separates w from the rest of G). Applying Proposition 10.5 to this graph after replacing L with $L' := L - vz \cup \{vw, wz\}$ we deduce that $\vec{wz}, \vec{wv}$ lie in the same L'-component of $\vec{L'^D}$, and it is straightforward to deduce that $\vec{vz}, \vec{zv}$ lie in the same L-component of $\vec{L^D}$ from this. $\square$

Next, we prove one of the desired properties of $\mathcal{P}$-interfaces, namely that $I_V \cup I_O$ spans a connected subgraph of G.

Proposition 10.7. *For every $\mathcal{P}$-interface $I = (I_V, I_O)$ of G, the edge-set I_O spans a connected subgraph of G incident with all edges in I_V, unless $I_O = \emptyset$ (in which case I_V is the set of edges incident with o).*

Proof. Let D be defined as in (ii) of Definition 10.3. By (iv) of Definition 10.3, for every $e \in I_O$ there is a $\mathcal{P}$-path P in $G - I_V$ connecting e to the head of an element of $\vec{I_V^D}$. Note that all edges of P belong to I_O as we can apply item (iv) to any of them, where we use the fact that since P meets D, it is contained in D because D is a component of $G - I_V$. This means that every component of the graph $G_O \subseteq G$ spanned by the edges in I_O contains the head of an element of $\vec{I_V^D}$.

Therefore, if G_O has more than one components, then these components define a proper bipartition (J_1, J_2) of $\vec{I_V^D}$, by letting J_1 be the set of all $j \in \vec{I_V^D}$ such that $head(j)$ lies in one of these components. Applying Definition 10.1 to this bipartition we obtain a contradiction, since for any $\mathcal{P}$-path P in $G - I_V$ connecting $j_1 \in J_1$ to $j_2 \in J_2$, all edges of P lie in I_O by the above remark, which implies that the heads of j_1 and j_2 lie in the same component of G_O. This proves that G_O is connected as claimed.

Finally, if some $e \in I_V$ is not incident with G_O, then we can apply the same argument to the bipartition of $\vec{I_V^D}$ one partition class of which consists of the one or two directions of e that lie in $\vec{I_V^D}$ (recall the remark after (iii) of Definition 10.3). If $I_O \neq \emptyset$, then this bipartition is proper because each component of G_O is incident with an element of $\vec{I_V^D}$ as we have proved, and we obtain a contradiction as above.

If $I_O = \emptyset$, and there are at least two vertices x, y of D incident with I_V, then we obtain a proper bipartition of I_V by letting one of the classes be the set of edges incident with x, say, and reach a contradiction with the same arguments. Thus all edges of I_V are incident with a vertex x of D in this case, and in order to satisfy (i) I_V must be the set of edges incident with $x = o$. $\square$

We have now gathered enough tools to prove our main result about $\mathcal{P}$-interfaces.

Proof of Theorem 10.4. **Existence:** To begin with, given such a cluster C we will find an occurring $\mathcal{P}$-interface (I_V, I_O) such that $I_O \subseteq E(C)$ and $I_V \subseteq \partial C$. For this, let

$$B := \{e \in \partial C \mid \text{ there is a path from } e \text{ to } \infty \text{ in } G - \partial C\}.$$

(This is the minimal subset of ∂C separating C from infinity.)

Fix an enumeration of the elements of $\vec{B^C}$ (this notation was introduced before Definition 10.3), and let $X_i, 1 \leq i \leq |\vec{B^C}|$ be the ∂C-component of $\overleftrightarrow{\partial C}$ containing the ith element of $\vec{B^C}$ in that enumeration (the definition of F-components is given after Definition 10.1). It will turn out that these components X_i coincide with each other, but we cannot use this fact yet. Let $J := \bigcup_i X_i$, and let I_V be the corresponding undirected edges, that is, $I_V := \{vw \in \partial C \mid \vec{vw} \in J\}$.

We will start by proving that I_V satisfies properties (i), (ii) and (iii), after which we can define I_O via (iv) to ensure that (I_V, I_O) is indeed a $\mathcal{P}$-interface.

To see that (i) is satisfied, we recall that $B \subseteq I_V$ by the definitions, and we claim that B separates o from infinity. This is true because if Q is an infinite path starting at o, then it has to contain an edge in ∂C by our assumption that ∂C separates o from infinity. The last such edge of Q then lies in B by the definitions. Thus all paths from o to infinity meet B, proving that (i) is satisfied.

It is easy to see that (ii) is satisfied by letting D be the component of $G - I_V$ containing C, which exists since $I_V \subseteq \partial C$. Indeed, $C \subseteq D$ meets all edges in ∂C, hence all edges in I_V.

We will now check that $\vec{I_V^D}$ is I_V-connected, that is, (iii) is satisfied. Proposition 10.5 —applied with $L = I_V$, so that D meets all edges in $B \subseteq I_V$ as remarked above— yields that $\vec{B^D}$ is contained in some I_V-component X of $\vec{I_V^D}$. We will prove that X contains the other edges of $\vec{I_V^D}$ too. For this, recall that X_i is a ∂C-component of $\overleftrightarrow{\partial C}$, and so X_i is ∂C-connected by the definition of ∂C-components. We can reformulate this by saying that X_i is (contained in) a ∂C-component of X_i. Recall that $J = \bigcup_i X_i$. Using (28) with $F = \partial C$ we will show that $J \subseteq \vec{I_V^D}$. Indeed, the component C of $G - \partial C$ contains the head of an element of X_i in $\vec{B^C}$ by the definition of X_i, and so the head of every element of J lies in C by (28). Since $C \subseteq D$, we deduce $J \subseteq \vec{I_V^D}$. Plugging these facts into Proposition 10.2 —with $Y = X_i$— we obtain that X_i is contained in an I_V-component of $\vec{I_V^D}$, because $X_i \subseteq \vec{I_V^D}$ and $I_V \subseteq \partial C$. Since each X_i meets $\vec{B^D}$, which is contained in the I_V-component X, (27) yields that X contains $J = \bigcup_i X_i$.

To conclude that $\vec{I_V^D}$ is I_V-connected, or in other words, that $X = \vec{I_V^D}$, it remains to show that if $e \in \vec{I_V^D} - J$ then e lies in X as well. To see this, note that for any such $e = \vec{vz}$ the reverse direction $e' := \vec{zv}$ lies in J, because all edges of I_V have at least one of their directions in J by the definitions. Moreover, we have $z, v \in V(D)$ since $e, e' \in \vec{I_V^D}$, where we used the fact that $J \subseteq \vec{I_V^D}$. Thus Proposition 10.6 —with $L = I_V$— yields that e, e' lie in a common I_V-component of $\vec{I_V^D}$. Using (27) again, combined with the fact that $(e' \in) J \subseteq X$ proved above, we deduce that $e \in X$ as desired. To summarize, we have proved that all elements of $\vec{I_V^D}$ lie in a common I_V-component X, in other words, $\vec{I_V^D}$ is I_V-connected, establishing (iii).

We proved above that $J \subseteq \vec{I_V^D}$. Next, we claim that actually $\vec{I_V^D} = J$, which will be used below. Suppose this is not the case, and consider the proper bipartition $(J, \vec{I_V^D} - J)$ of $\vec{I_V^D}$. Since $\vec{I_V^D}$ is I_V-connected, there is a $\mathcal{P}$-path P in $G - I_V$ connecting directed edges $e \in J$ to $f \in \vec{I_V^D} - J$. Let g be the first edge of P that lies in ∂C, directed towards e, if such an edge exists, and let $g = f$ otherwise. In both cases, the subpath P' of P from e to g avoids ∂C, and hence proves that e and g lie in a common ∂C-component of $\overleftrightarrow{\partial C}$. But then g must lie in J since J is a union of ∂C-components of $\overleftrightarrow{\partial C}$. This contradicts that $g \notin J$ when $g = f$ and $g \notin I_V$ otherwise. This contradiction proves that $\vec{I_V^D} = J$.

Thus using (iv) of Definition 10.3 to define I_O, we obtain a $\mathcal{P}$-interface $I := (I_V, I_O)$. Since $I_V \subseteq \partial C$ which is vacant, to show that I occurs it remains to show that I_O is occupied in ω. This is true because if P is a $\mathcal{P}$-path in $G - I_V$ connecting some edge e of I_O to $\vec{I_V^D} = J$, then the last vacant edge f of the extended path $\{e\} \cup P$, if such an edge f exists, would have to lie in I_V by the definitions and the fact that $\vec{I_V^D} = J$, contradicting that $\{e\} \cup P$ avoids I_V. Hence no such f exists, and in particular any $e \in I_O$ is occupied as desired. Moreover, I meets C because $I_O \cup I_V$ spans a connected subgraph of G by Proposition 10.7, and that subgraph contains B, hence meets C.

To prove the claim that $I_O \subseteq E(C)$, recall that I_O spans a connected subgraph G_o of G by Proposition 10.7. This subgraph meets C unless it is empty, because G_o is incident with all of $I_V \supseteq B$, and it cannot meet the infinite component of $G - B$ as it is contained in D. Since I_O, being occupied, avoids ∂C, we deduce that $I_O \subseteq E(C)$ indeed.

Uniqueness: Suppose that our cluster C is met by a further occurring $\mathcal{P}$-interface $I' = (I'_V, I'_O) \neq I$. By Lemma 10.7 the subgraph of G spanned by $I'_O \cup I'_V$ is connected, and therefore contained in $C \cup \partial C$ since I'_O meets $E(C)$. It follows that $I'_V \subseteq \partial C$ since I' occurs.

Let D' be the component of $G - I'_V$ defined in (ii). We claim that $B \subset I'_V$. Indeed, if I'_V misses some edge of B, then $I'_V \subseteq \partial C$ does not separate C from infinity, hence $C \cap D' = \emptyset$, contradicting that $I'_O \subseteq E(D')$ and $I'_O \cap E(C) \neq \emptyset$ unless $E(C) = \emptyset$, in which case I'_V cannot separate o from infinity violating (i).

Moreover, we have $D' \supseteq C$ since $I'_V \subseteq \partial C$ (because I' occurs) and I'_O meets $E(C)$.

We will first prove that $I'_V \subseteq I_V$. So let $f \in I'_V$, and suppose for a contradiction that $f \notin I_V$. In this case, the bipartition $(J, J' := \overleftrightarrow{\partial C} - J)$ of $\overleftrightarrow{\partial C}$, where J is as in the definition of I_V in the existence part, is such that $\vec{B}^C \subseteq J$ and both directions $\vec{f}, \overleftarrow{f}$ of f lie in J' and there is no $\mathcal{P}$-path in $G - \partial C$ connecting J to J'.

Consider now the bipartition $(J \cap I'^{\vec{D}'}_V, J' \cap I'^{\vec{D}'}_V)$ of $I'^{\vec{D}'}_V$, which is proper because $\vec{B}^C \subseteq I'^{\vec{D}'}_V$ (because $D' \supseteq C$ and $B \subset I'_V$) and $\{\vec{f}, \overleftarrow{f}\} \cap I'^{\vec{D}'}_V \neq \emptyset$ (by the definition of D'). Therefore, since $I'^{\vec{D}'}_V$ is I'_V-connected by (iii), there is a $\mathcal{P}$-path P in $G - I'_V$ connecting $J \cap I'^{\vec{D}'}_V$ to $J' \cap I'^{\vec{D}'}_V$. Let e be the last edge of P in ∂C, which exists because P cannot avoid ∂C by the aforementioned property of the bipartition (J, J'), and let P' be the final subpath of P starting at e. But then applying (iv) to I' using the path P' we deduce that $e \in I'_O$, contradicting that I' occurs and $e \in \partial C$ is vacant. This contradiction proves that $I'_V \subseteq I_V$.

Next, we prove that $I_V \subseteq I'_V$ as well. Indeed, if $I_V \not\subseteq I'_V$, then the bipartition $(I^{\vec{D}}_V \cap \overleftrightarrow{I'_V}, I^{\vec{D}}_V - \overleftrightarrow{I'_V})$ of $I^{\vec{D}}_V$ is proper because $\vec{B}^C \subseteq I^{\vec{D}}_V, \overleftrightarrow{I'_V}$. Since $I^{\vec{D}}_V$ is I_V-connected, there is a $\mathcal{P}$-path P in $G - I_V$ connecting some edge $f \in I'_V$ to some edge $e \in I_V - I'_V$. Since we have proved that $I'_V \subseteq I_V$, we deduce that P lies in $G - I'_V$. But then applying (iv) to I' using the path P we deduce that $e \in I'_O$, contradicting that I' occurs and $e \in I_V \subseteq \partial C$ is vacant. This contradiction proves that $I_V \subseteq I'_V$, and hence $I'_V = I_V$.

To conclude that I is the unique occuring $\mathcal{P}$-interface that meets C, it remains to prove that $I'_O = I_O$. But this is now obvious from (iv), since $I'_V = I_V$ and hence $D' = D$ by (ii).

Converse: Suppose now that (I_V, I_O) is a $\mathcal{P}$-interface occurring in a percolation instance ω. Then by Lemma 10.7 it meets a unique cluster C of ω, and we have $I_V \subseteq \partial C$ by what we proved above. By (i) I_V, and hence ∂C, separates o from infinity. $\square$

10.5 Using $\mathcal{P}$-interfaces to prove analyticity

Define the *boundary size* of a $\mathcal{P}$-interface $I = (I_V, I_O)$ to be $|I_V|$. Note that every set S of edges which is S-connected (according to Definition 10.1) corresponds to a connected induced subgraph of the mth power of the line graph $L(G)^m$ of G, where $m = m_{\mathcal{P}} := \lfloor t/2 \rfloor$ and t is the length of the longest cycle in $\mathcal{P}$. The degree of each vertex of $L(G)$ is at most $2d-2$, where d is the maximum degree of G (we are still assuming that G satisfies (29)), and so the degree of each vertex of $L(G)^m$ is at most $(2d-2)^m$. Applying the remark after Proposition 14.1 to $L(G)^m$, combined with the fact that any $\mathcal{P}$-interface $I = (I_V, I_O)$ is uniquely determined by I_V by the definitions, we thus deduce that

Lemma 10.8. *The number of $\mathcal{P}$-interfaces (I_V, I_O) of G of boundary size n such that I_V contains a fixed edge of G is less than $c\gamma_{\mathcal{P}}^n$, where c is a constant, $\gamma_{\mathcal{P}} = ((2d-2)^{m_{\mathcal{P}}} - 1)e$, and d is the maximum degree of G.*

The following is the analogue of Proposition 7.5.

Lemma 10.9. *For every $\mathcal{P}$-interface $I = (I_V, I_O)$ of G, we have $|I_V| \geq |I_O|/d^t$, where d is the maximum degree of G and t is the length of the longest cycle in $\mathcal{P}$.*

Proof. By (iv) of Definition 10.3, each $e \in I_O$ has distance less than t from I_V in the subgraph G_I of G spanned by $I_V \cup I_O$. Using this fact we can assign each $e \in I_O$ to an edge $f(e)$ of I_V so that the distance between e and $f(e)$ in G_I is less than t. Then the number $|f^{-1}(g)|$ of edges of I_O assigned to any $g \in I_V$ is at most the size of the ball of radius $t-1$ around g in G, which is at most d^{t-1} since G is d-regular. Thus $|I_V| \geq |I_O|/d^{t-1}$ by the pigeonhole principle. $\square$

Let $R = \dots, r_{-1}, r_0, r_1, \dots$ be 2-way infinite geodesic with $r_0 = o$ (such a geodesic exists in every Cayley graph by an elementary compactness argument, provided we assume e.g. the Axiom of Countable Choice). Let f_i denote the edge $r_i r_{i+1}$ of R.

Lemma 10.10. *For every $\mathcal{P}$-interface $I = (I_V, I_O)$ of o with boundary size $|I_V| = n$, the set I_V contains at least one of the edges $f_0, f_1, \dots, f_{d^t n-1}$.*

Proof. Each of the two 1-way infinite subpaths of R starting at o connects o to infinity, so I_V must contain an edge from each of them. By Proposition 10.7 and Proposition 10.9, I_O is connected, incident to both of these edges, and $|I_O| \leq d^t n$. Thus if I_V contains some edge $r_i r_{i+1}$ with $i \geq d^t n$, then it cannot meet $\dots r_{-1} r_0$ because R is a geodesic. $\square$

A multi-$\mathcal{P}$-interface S is a finite set of $\mathcal{P}$-interfaces $\{(I_V^i, I_O^i)\}_{1 \leq i \leq k}$ such that the corresponding graphs G_O^i, i.e. the subgraphs of G spanned by the edges in I_O^i, are pairwise vertex disjoint. Define the *boundary* ∂S of S to be $\bigcup_{1 \leq i \leq k} |I_V^i|$. Let $\mathcal{MS}$ denote the set of multi-$\mathcal{P}$-interfaces and $\mathcal{MS}_n$ the set of multi-$\mathcal{P}$-interfaces of total boundary size n. Using the above lemma and Proposition 10.10 we can upper bound the number of elements of $\mathcal{MS}_n$ that can occur simultaneously in any ω similarly to the proof of Proposition 7.7.

Lemma 10.11. *There is a constant $x \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ at most $x^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any ω.*

We will now prove that $p_{\mathbb{C}} < 1$ for every finitely presented Cayley graph following the approach of Section 7.2, replacing the use of exponential decay of the dual by Lemmas 10.8 and 10.9.

Theorem 10.12. *Let G be an 1-ended Cayley graph with a finite presentation $\mathcal{P}$. Then $p_{\mathbb{C}} \leq 1 - 1/\gamma_{\mathcal{P}}$ for bond percolation on G.*

Proof. Similarly to (11), we claim that

$$1 - \theta_o(p) = \sum_{S \in \mathcal{MS}} (-1)^{c(S)+1} Q_S(p) \tag{30}$$

for every $p \in (q, 1]$, where $c(S)$ denotes the number of $\mathcal{P}$-interfaces in the multi-$\mathcal{P}$-interface S, and $Q_S(p) := \mathbb{P}_p(S \text{ occurs})$.
We will use Proposition 10.4 to prove that the above formula holds. By that proposition, $C(o)$ is finite if and only if it meets a $\mathcal{P}$-interface. Since for any pair of distinct occuring $\mathcal{P}$-interfaces the graphs G_O do not share a vertex, the inclusion-exclusion principle yields

$$1 - \theta_o(p) = \mathbb{P}(\text{ at least one } \mathcal{P}\text{-interface occurs }) = \sum_{S \in \mathcal{MS}} (-1)^{c(S)+1} Q_S(p)$$

provided the latter sum converges absolutely.

Once again

$$\sum_{S \in \mathcal{MS}_n} Q_S(p) = \mathbb{E}_p(\sum_{S \in \mathcal{MS}_n} \chi_{\{S \text{ occurs}\}})$$

and by Proposition 10.11 we conclude that

$$\sum_{S \in \mathcal{MS}_n} Q_S(p) \leq x^{\sqrt{n}} \mathbb{P}_p(\text{some } S \in \mathcal{MS}_n \text{ occurs}).$$

The event $\{\text{some } S \in \mathcal{MS}_n \text{ occurs}\}$ implies that a set of edges with certain properties is vacant and our goal is to use Peierls' argument to conclude that the probability of the latter event decays exponentially for large enough p.

Let $S \in \mathcal{MS}_n$ and let $X_1, X_2, \ldots, X_k$ be the components of the subgraph of $L(G)^m$ spanned by ∂S, where $m = \lfloor t/2 \rfloor$. By the argument at the beginning of Section 10.5, each X_i contains the boundary of a $\mathcal{P}$-interface of size at most $n_i := |X_i|$. Thus by Proposition 10.10, X_i contains one of the edges of $f_0, f_1, \ldots, f_{d^t n_i - 1}$. The Hardy–Ramanujan formula and Proposition 10.8 now easily yield that the number of all possible boundaries of $\mathcal{MS}_n$ is at most

$$r^{\sqrt{n}} \max\{c^k d^{kt} n_1 n_2 \ldots n_k\} \gamma_{\mathcal{P}}^n,$$

where the maximum ranges over all partitions $\{n_1, n_2, \ldots, n_k\}$ of n such that every N appears at most $d^t N$ times. As in the proof of Theorem 7.1, it is easy to check that the quantity $\max\{c^k d^{kt} n_1 n_2 \ldots n_k\}$ grows subexponentially in n. Since each $S \in \mathcal{MS}_n$ occurs with probability at most $(1-p)^n$ by the definitions, we conclude that

$$\mathbb{P}_p(\text{some } S \in \mathcal{MS}_n \text{ occurs}) \leq r^{\sqrt{n}} \max\{c^k d^{kt} n_1 n_2 \ldots n_k\} \gamma_{\mathcal{P}}^n (1-p)^n, \tag{31}$$

and thus $\mathbb{P}_p(\text{some } S \in \mathcal{MS}_n \text{ occurs})$ decays exponentially for every $p > 1 - 1/\gamma_{\mathcal{P}}$.

Finally, combining this exponential decay with Proposition 4.14 and Proposition 10.9 we deduce that θ is analytic in $(1 - 1/\gamma_{\mathcal{P}}, 1]$, arguing as in the end of the proof of Theorem 7.1. $\square$

Proposition 10.12 immediately implies

Corollary 10.13. *For $G = \mathbb{Z}^d$ we have $p_{\mathbb{C}} \leq 1 - 1/\gamma_d$ where $\gamma_d = ((4d-2)^2 - 1)e$.*

Here, $\mathbb{Z}^d$ denotes the cubic lattice in $\mathbb{R}^d$. That $p_{\mathbb{C}} < 1$ for $G = \mathbb{Z}^d$ is also proved in [16] and [15]; our bounds are better than those of [16] and worse than those of [15]. They could be improved if we had a more precise upper bound on the number of $\mathcal{P}$-interfaces in this case than the one provided by Lemma 10.8. Our proof that $p_{\mathbb{C}} < 1$ can be extended to any quasi-transitive lattice in $\mathbb{R}^d$. For this we need to show that such a graph admits a basis of its cycle space with bounded cycle lengths, but this is just an exercise.

10.6 Extending to site percolation

In this section we extend Theorem 10.15 to site percolation. The proof is essentially the same, all we have to do is to adapt the probability $(1-p)^n$ appearing in (31), but we will also adapt Lemma 10.8 in order to obtain a better bound on $p_{\mathbb{C}}$.

For a $\mathcal{P}$-interface (I_V, I_O) of G we let V_O denote the set of vertices incident with an edge in I_O, and we let V_V denote the set of vertices incident with an edge in I_V but with no edge in I_O. We say that a $\mathcal{P}$-interface $I = (I_V, I_O)$ is a *site-$\mathcal{P}$-interface*, if no edge in I_V has both its end-vertices in V_O. Note that any site percolation instance $\omega \in \{0,1\}^{V(G)}$ naturally gives rise to a bond percolation instance $\omega' \in \{0,1\}^{E(G)}$, by setting $\omega'(xy) = 1$ whenever $\omega(x) = 1$ and $\omega(y) = 1$. It is obvious from the definitions that if I occurs in such an ω', then I is a site-$\mathcal{P}$-interface. For site-$\mathcal{P}$-interfaces we can improve Lemma 10.8 as follows, using the same proof except that we work with G rather than $L(G)$. The *vertex-boundary size* of (I_V, I_O) is $|V_V|$.

Lemma 10.14. *The number of site-$\mathcal{P}$-interfaces (I_V, I_O) of G of vertex-boundary size n such that I_V contains a fixed edge of G is less than $c'\dot{\gamma}_{\mathcal{P}}^n$, where $\dot{\gamma}_{\mathcal{P}} = (d^{m_{\mathcal{P}}} - 1)e$, and d is the degree of G.*

Using this we can now adapt Theorem 10.12 to site percolation, repeating the proof verbatim, except that we use site-$\mathcal{P}$-interfaces instead of $\mathcal{P}$-interfaces.

Corollary 10.15. *Let G be an 1-ended Cayley graph with a finite presentation $\mathcal{P}$. Then $p_{\mathbb{C}} \leq 1 - 1/\dot{\gamma}_{\mathcal{P}}$ for site percolation on G.*

This bound on $p_{\mathbb{C}}$ is far from the conjectured $p_{\mathbb{C}} = p_c$, but not so far from $p_{\mathbb{C}} \leq 1 - p_c$, which is the best that our methods can achieve (and possibly the truth) in light of a result of Kesten & Zhang, saying that for site percolation on $\mathbb{Z}^d, d \geq 3$, the distribution of the vertex-boundary size of the site-$\mathcal{P}$-interface of the cluster of the origin does not have an exponential tail [39, Theorem 4] (here $\mathbb{Z}^d$ denotes the cubic lattice in $\mathbb{R}^d$, and the basis $\mathcal{P}$ consists of the squares bounding the faces of its cubes). Our next result implies that this 'theoretical barrier' $p_{\mathbb{C}} \leq 1 - p_c$ can in fact be achieved if we are allowed to modify the graph a little by adding some diagonal edges.

Theorem 10.16. *Let G be an 1-ended quasi-transitive graph admitting a basis $\mathcal{P}$ of $\mathcal{C}(G)$ all cycles of which are triangles. Then $p_{\mathbb{C}} \leq 1 - p_c$ for both site and bond percolation on G. In particular, we have $p_c \leq 1/2$ unless $p_c = 1$.*

In particular, we can obtain such a G by adding to $\mathbb{Z}^d$ the 'monotone' diagonal edges, i.e. the edges of the form xy where $y_i - x_i = 1$ for exactly two coordinates $i \leq d$, and $y_i = x_i$ for all other coordinates. Then each square gives rise to two triangles, and we can use all these triangles as our basis $\mathcal{P}$ of the cycle space.

Note that for $d = 2$ we obtain the triangular lattice, and so Theorem 10.16 can be thought of as a generalisation of Corollary 7.9.

For its proof we will need the following lemma, which is a special case of [51, THEOREM 5.1], the main idea of which we used in Proposition 10.5, as illustrated in Figure 5.

Lemma 10.17. *Let G be an 1-ended quasi-transitive graph admitting a basis $\mathcal{P}$ of $\mathcal{C}(G)$ all cycles of which are triangles. Then for every site-$\mathcal{P}$-interface (I_V, I_O) of G, the vertex boundary V_V spans a connected subgraph of G.*

Proof of Theorem 10.16. We first prove the statement for site percolation. We follow the lines of the proof of Theorem 7.1, except that we now let $\mathcal{MS}_n$ denote the set of multi-$\mathcal{P}$-interfaces all elements of which are site-$\mathcal{P}$-interfaces. Instead of Lemma 7.6, which states that the boundary of a $\mathcal{P}$-interface spans a connected subgraph of the dual lattice in that setup, we now use Lemma 10.17, which is the analogous statement for the boundary V_V of a site-$\mathcal{P}$-interface under our assumption on $\mathcal{P}$ that all its cycles are triangles. The proof of Lemma 7.7 can be repeated verbatim, except that we replace the quasi-geodesic X used there with an arbitrary 2-way infinite quasi geodesic of G, which exists by a standard compactness argument. In that proof, we used the canonical coupling between bond percolation on a planar lattice and its dual, and applied the Aizenman-Barsky property to the subcritical clusters of the dual. Here, we instead use the canonical coupling between site percolation with parameter p and with parameter $1 - p$ obtained by switching between vacant and occupied vertices. We apply the Aizenman-Barsky property to the boundaries V_V of our site-$\mathcal{P}$-interfaces: since they span connected subgraphs of G by Lemma 10.17, each such V_V is contained in a cluster of vacant sites. But as $p > 1 - p_c$, vacant clusters are subcritical due to that coupling, hence their size distribution has an exponential tail by the Aizenman-Barsky property (Theorem 3.1). The rest of the proof can be repeated as is.

To prove the statement for bond percolation, we use the canonical coupling between bond percolation on G and site percolation on its line graph $L(G)$, noting that the cluster of an edge of G is infinite in the former if and only if the cluster of the corresponding vertex of $L(G)$ is infinite in the latter. Our plan is to apply the statement for site percolation we just proved to $L(G)$. Note that if G is quasi-transitive, then so is $L(G)$. Moreover, it is straightforward to check that we can obtain a basis of $\mathcal{C}(L(G))$ from any basis $\mathcal{P}$ of G by adding all the triangles of the form x, y, z in $L(G)$ whenever the edges x, y, z of G are incident with a common vertex. Thus we can reduce to the case of site percolation as desired.

For both site and bond percolation, since $p_c \leq p_{\mathbb{C}}$ unless $p_c = 1$ because $\theta(p)$ can never be analytic at p_c if $p_c < 1$, we immediately obtain $p_c \leq 1/2$. $\square$

11 Triangulations

11.1 Overview

In this section we use the techniques we developed to provide upper bounds on p_c and $\dot{p}_c$ for certain families of triangulations. Although these bounds will apply to $p_{\mathbb{C}}$, we stress that the results of this section give the best known (or only) such bounds on $p_c, \dot{p}_c$ for these triangulations.

We will prove that $p_{\mathbb{C}} \leq 1/2$ for Bernoulli bond percolation on triangulations of an open disk that either satisfy a weak expansion property or are transient. Once again the analyticity of θ_o will follow by showing that the outer-interfaces ($\mathcal{P}$-interfaces) of o have an exponential tail for every $p > 1/2$.

The interest in the study of percolation on triangulations of an open disk was sparked by the seminal paper [13] of Benjamini & Schramm. They made a series of conjectures, the strongest one of which is that $\dot{p}_c(T) \leq 1/2$ on any bounded degree triangulation T of an open disk that satisfies a weak isoperimetric inequality of the form $|\partial_V A| \geq f(|A|)) \log(|A|))$ for some function $f = \omega(1)$, where S is any finite set of vertices. More recently, Benjamini [12] conjectured that $\dot{p}_c(T) \leq 1/2$ on any transient bounded degree triangulation T of an open disk.

Angel, Benjamini & Horesh [6] proved that for any triangulation T of an open disk with minimum degree 6, the isoperimetric dimension of T is at least 2 and thus satisfies the assumption of the conjecture of Benjamini & Schramm. They also asked whether $p_c(T) \leq 2\sin(\pi/18)$ (and $\dot{p}_c \leq 1/2$), the critical value for bond percolation on the triangular lattice, for any such triangulation.

The main results of this section, which we now state, imply that in all aforementioned conjectures, the bound $p_c \leq 1/2$ is correct if one considers bond instead of site percolation.

Theorem 11.1. *Let T be a triangulation of an open disc such that every vertex has finite degree (not necessarily bounded) and*[6]

for all but finitely many sets A of vertices we have $|\partial_V A| \geq k \log(diam(A))$ for some constant $k > 0$. (32)

Then there is a constant $\nu_k < 1$ that converges to $1/2$ as k goes to infinity, such that

$$p_c(T) \leq p_{\mathbb{C}}(T) \leq \nu_k.$$

In particular, if

for every finite set A of vertices we have $|\partial_V A| \geq f(diam(A)) \log(diam(A))$ for some function $f = \omega(1)$, (33)

then

$$p_c(T) \leq p_{\mathbb{C}}(T) \leq 1/2.$$

(This holds in particular when $h(T) > 0$, i.e. when T is non-amenable.)

Theorem 11.2. *Let T be a transient triangulation of an open disc with degrees bounded above by d. Then*

$$p_c(T) \leq p_{\mathbb{C}}(T) \leq 1/2.$$

[6]The reader will lose nothing by replacing $diam(A)$ by $|A|$ in this statement, which only strengthens our assumptions.

We will also prove the same bounds for recurrent triangulations T with a uniform upper bound on the radii of the circles in any circle packing of T, as well as analogues for site percolation (Section 11.3).

11.2 Proofs

Notice that any bounded degree triangulation T satisfies the assumptions (29) of Section 10.4. Hence the arguments of that section imply that $p_{\mathbb{C}}(T) < 1$ for bond percolation provided we further assume that T contains a 2-way infinite geodesic. However, the latter is a rather strong condition. But we only used the existence of a 2-way infinite geodesic in the proof of Lemma 10.10, and it will turn out that a variant of that lemma still holds for transient triangulations and triangulations satisfying the above isoperimetric inequality.

We will first focus on proving Theorem 11.1, but many of the following arguments will also be valid for transient triangulations.

Our proofs will follow the lines of that of Theorem 7.1. Recall the definitions of outer-interface and multi-interface of Section 7. Again $\mathcal{MS}$ denotes the set of multi-interfaces of a chosen vertex o, while ∂M denotes the boundary of a multi-interface M and $\mathcal{MS}_n := \{M \in \mathcal{MS} \mid |\partial M| = n\}$.

Let T be a triangulation of an open disk and o a vertex in T. Once again we will utilise the inclusion-exclusion principle to express $1 - \theta_o$ as an infinite sum

$$1 - \theta_o(p) = \textstyle\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} Q_M(p) \tag{34}$$

for every p large enough, where $c(M)$ denotes the number of outer-interfaces in the multi-interface M, and $Q_M(p) := \mathbb{P}_p(M \text{ occurs})$. The validity of the formula will follow as in the proof of Theorem 7.1 (recall (11)) once we establish an exponential tail for the corresponding probabilities, which is the purpose of the following lemma.

Lemma 11.3. *There is a constant $\nu_k < 1$ that converges to $1/2$ as k goes to infinity, such that for every triangulation T of an open disk satisfying condition* (32) *of Theorem 11.1 and every $p \in (\nu_k, 1]$,*

$$\textstyle\sum_{M \in \mathcal{MS}_n} Q_M(p) \leq c_1 {c_2}^n, \tag{35}$$

where $c_1 = c_1(p) > 0$ and $c_2 = c_2(p) > 0$ are some constants with $c_2 < 1$. Moreover, if $[a, b] \subset (\nu_k, 1]$, then the constants c_1 and c_2 can be chosen independent of p in such a way that (35) *holds for every $p \in [a, b]$.*

In order to prove the above lemma, we first pick an arbitrary infinite geodesic R starting from o. Our goal is to show that the outer-interfaces M of o for which ∂M contains a fixed edge $e \in E(R)$, occur with exponentially decaying probability for every large enough value of p. Then we will upper bound the choices for $e \in R$.

In what follows we will be using the standard coupling between percolation on T and its dual T^* as in the proof of Lemma 7.8. Since T is a triangulation, the dual of any minimal cut of T is a cycle. The number of cycles in T^* of size n containing a fixed edge is at most 2^{n-1}, because T^* is a cubic graph. Then the union bound shows that the probability that some minimal cut containing a fixed edge is vacant has an exponential tail for every $p > 1/2$. However, the boundary of an outer-interface is not necessarily a minimal cut. Still, the dual of the boundary of any outer-interface in T is a connected subgraph of T^*. The desired exponential tail will follow from the Aizenman-Barsky property (Theorem 3.1) once we show that $\sup_{u\in V(T^*)}\chi_u(p) < \infty$ for every $p < 1/2$, where as usual $\chi_u(p)$ denotes the expected size of the percolation cluster of u. The next lemma proves the this statement.

Lemma 11.4. *Let T be a triangulation of an open disc. Then*

$$\chi^*(p) := \sup_{u\in V(T^*)} \chi_u(p) < \infty$$

for every $p \in (1/2, 1]$.

Proof. Let u be a vertex of T^*. Note that whenever some vertex v belongs to $C(u)$ there is a self-avoiding walk from u to v with occupied edges. Hence we obtain $\mathbb{E}(|C(u)|) \leq \mathbb{E}_p(P(u))$, where $P(u)$ is the number of occupied self-avoiding walks starting from x. The number $\sigma_k(u)$ of k-step self-avoiding walks in T^* starting from u is at most $3 \cdot 2^{k-1}$. Consequently,

$$\mathbb{E}_{1-p}(P(u)) \leq \textstyle\sum_{k=0}^{\infty} 3 \cdot 2^{k-1}(1-p)^k < \infty \tag{36}$$

whenever $p > 1/2$. Since this bound does not depend on u the proof is complete. □

Using Theorem 3.1 we immediately obtain the desired exponential tail.

Corollary 11.5. *For every $p > 1/2$ there is a constant $0 < c = c(p) < 1$ such that for any triangulation T of an open disk and any vertex $u \in T^*$, we have $\mathbb{P}_{1-p}(|C(u)| \geq n) \leq c^n$.*

The following lemma converts condition (32) into a statement saying that every outer-interface of T meets a relatively short initial subpath of R.

Lemma 11.6. *Let T be a triangulation of an open disk satisfying condition (32). Let R be a geodetic ray in T starting at any $o \in V(G)$, and let R_n be the set of edges of R contained in some outer-interface of $\mathcal{S}_n$. Then $|R_n| \leq e^{n/k}$ for all but finitely many values of n.*

Proof. Define a function $g : \mathbb{N} \to \mathbb{N}$ by letting $g(n)$ be the smallest integer l such that every outer-interface of $\mathcal{S}_n$ contains at least one of the first l edges of R if such a l exists, and let $g(n) = \infty$ otherwise, with the convention that $g(n) = 1$ if no such edge-separator of size n exists.

We need to show that $g(n) \leq e^{n/k}$ for almost every n (in particular, $g(n) < \infty$). In other words, we need to show that $g(n) > e^{n/k}$ holds for only finitely many values of n. To see this, assume n is such a number, which means that some outer-interface M of $\mathcal{S}_n$ does not contain any of the first $e^{n/k}$ edges of R. Let B be the minimal cut of M and $A = A_n$ be the component of o in $G - B$. Our condition (32) says that

$$k \log(diam(A)) \leq |\partial_V A| \leq |\partial_E A| = |B| \leq n,$$

except possibly for finitely many sets $A = A_n$, hence for finitely many values of n.

On the other hand, we have $diam(A) > e^{n/k}$ since A contains the first $e^{n/k}$ edges of the geodesic R. Combining these inequalities yields the contradiction $k \log e^{n/k} > n$.

□

An immediate consequence of Lemma 11.6 is that $g(n)$ grows subexponentially in n, i.e. $\limsup_{n\to\infty} g(n)^{1/n} = 1$, whenever the stronger condition (33) is satisfied.

Note that the constant $e^{-1/(2\chi^2)}$ involved in the statement of the theorem of Aizenman & Barksy does not converge to 0 as p goes to 0, because $\chi \geq 1$. Hence when we combine Corollary 11.5 and Lemma 11.6 with the union bound, we deduce that $\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs})$ decays exponentially in n for every large enough value of p, only when T satisfies (32) for some large enough value of k. In particular, when T satisfies (33), then $\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs})$ decays exponentially in n for every $p > 1/2$.

To cover the remaining cases, we will prove in the next lemma an exponential upper bound for the number of all possible multi-interfaces of $\mathcal{MS}_n$ and then we will deduce the desired exponential decay using a Peierls type argument.

A straightforward application of Corollary 14.1 yields

Lemma 11.7. *For every graph G with maximum degree 3, and any vertex $e \in E(G)$, the number of 2-connected subgraphs of G with m edges containing e is at most ν^m for some constant ν.*

The following lemma is the analogue of Lemma 7.7.

Lemma 11.8. *There is a constant $r \in \mathbb{R}$ such that for every triangulation T of an open disk satisfying condition (32) of Theorem 11.1 and every $n \in \mathbb{N}$ at most $tr^{\sqrt{n}}e^{n/k}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any percolation instance ω, where $t = t(T,k) > 0$ is a constant depending on T and k.*

Proof. Let S be an element of $\mathcal{MS}_n$, comprising the outer-interfaces $S_1, S_2, \ldots, S_l$. Since any two distinct occurring outer-interfaces are vertex disjoint by Lemma 7.4, the sizes m_i of ∂S_i define a partition of n. We call the multiset $\{m_1, m_2, \ldots, m_l\}$ the *boundary partition* of S. It is possible that more than one occuring multi-interfaces have the same boundary partition. In order to prove the desired assertion we will show that for every partition $\{m_1, m_2, \ldots, m_l\}$ of n the number of occuring multi-interfaces with $\{m_1, m_2, \ldots, m_l\}$ as their boundary partition is at most $te^{n/k}$ for some constant $t > 0$. Then the assertion follows by the Hardy–Ramanujan formula (Theorem 3.3).

Since occurring outer-interfaces meet R and they are vertex-disjoint by Lemma 7.4, S is uniquely determined by the subset of R it meets. We can utilise Lemma 11.6 to conclude that the number of occurring outer-interfaces with boundary of size m_i is at most $e^{m_i/k}$ for every $m_i \geq N$, where N is a sufficiently large positive integer. It is easy to see that the number of outer-interfaces with boundary of size at most N is bounded from above by some constant $M > 0$. Hence the number of occuring multi-interfaces with $\{m_1, m_2, \ldots, m_l\}$ as their boundary partition is bounded above by $Me^{n/k}$. □

We are now ready to prove Lemma 11.3.

Proof of Lemma 11.3. By Lemma 11.8 we have

$$\sum_{M \in \mathcal{MS}_n} Q_M(p) \leq tr^{\sqrt{n}}e^{n/k}\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs})$$

for every k. Let r_m denote the mth edge of R. We pick one of the two endpoints from every dual edge r_m^* and we denote it v_m (maybe some of these endpoints are the same). Let $D(m)$ denote the event that one of the clusters of $v_1, \ldots, v_{g(m)}$ contains at least m vertices. Arguing as in the proof of Lemma 7.8 we can deduce that

$$\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs}) \leq \sum_{\{m_1,\ldots,m_k\} \in P_n} \mathbb{P}_{1-p}(D(m_1)) \cdot \ldots \cdot \mathbb{P}_{1-p}(D(m_k)),$$

where P_n is the set of partitions of n. By Corollary 11.5 and the union bound we obtain

$$\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs}) \leq tr^{\sqrt{n}} e^{n/k} c^n,$$

where c is the constant of Corollary 11.5.

When k is large enough, there is some constant $\nu_k < 1$ such that $e^{2/k} c < 1$ for every $p > \nu_k$. This proves the exponential decay of $\sum_{M \in \mathcal{MS}_n} Q_M(p)$ when k is large enough. Moreover, as k goes to infinity, $e^{2/k}$ converges to 1 and thus it is easy to choose ν_k so that it converges to $1/2$.

For small values of k we can argue as in the proof of Theorem 10.12 to conclude that

$$\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs}) \leq tr^{\sqrt{n}} e^{n/k} \nu^n (1-p)^n,$$

where ν is a constant provided by Lemma 11.7. Hence $\sum_{M \in \mathcal{MS}_n} Q_M(p)$ decays exponentially in n for all $p > 1 - 1/\nu e^{2/k}$. □

The following is an easy combinatorial exercise.

Lemma 11.9. *For every triangulation of a disk T and every outer-interface M we have $|E(M)| \leq 2|\partial M|$.*

Proof. Let H be a finite connected graph witnessing the fact that M is an outer-interface. We claim that every edge $e \in M$ lies in a triangular face T_e of T such that at least one edge of $T_e - e$ lies in ∂M. Indeed, e lies in exactly two (triangular) faces of T, and we choose T_e to be one of them lying in the unbounded face of H; such a T_e exists, because by definition the vertices and edges of M are incident with the unbounded face of H. As T_e lies in the unbounded face of H, one of the two other edges of T_e lies in ∂M.

Since any edge of ∂M lies in at most two of these triangular faces T_e, the result follows. □

We have collected all the ingredients for the main result of this section.

Proof of Theorem 11.1. We first remark that $p_c < 1$ by (34) because, easily, $c_2(p) \to 0$ as $p \to 1$.

To obtain our precise bounds, note that, by definition, every $M \in \mathcal{MS}_n$ has n vacant edges. Moreover, $|E(M)| \leq 2n$ by Lemma 11.9. Hence we can now apply Corollary 4.14 for $I = (\nu_k, 1]$, $L_n = \mathcal{MS}_n$, and $(E_{n,i})$ an enumeration of the events $\{M \text{ occurs}\}_{M \in \mathcal{MS}_n}$, to deduce that $\theta_o(p)$ is analytic for $p > \nu_k$. As usual, we then recall that $\theta_o(p)$ cannot be analytic at p_c, and so $p_c \leq p_{\mathbb{C}}$. □

Remark: The above proof uses some complex analysis (needed in Corollary 4.14) to prove $p_c < 1/2$. But the complex analysis can be avoided by using a refinement of the Peierls argument that can be found in [46][Theorem 4.1].

For the proof of Theorem 11.2 we just need to show that the size of the set of edges of a 1-way geodesic R that meets $\bigcup \mathcal{MS}_n$ grows subexponentially in n. To this end, we will use the well known theorem of He & Schramm stating that every graph as in our statement is the contacts graph of a circle packing whose carrier is the open unit disc $\mathbb{D}$ in $\mathbb{R}^2$; see [36], where the relevant definitions can be found. We say that an edge e *meets* $\mathcal{MS}_n$, if there is $M \in \mathcal{MS}_n$ with $e \in \partial M$.

Lemma 11.10. *Let T be a triangulation of an open disk which is transient and has bounded vertex degrees. Let R be a geodetic ray in T starting at any $o \in V(G)$, and let R_n be the set of edges of R meeting $\mathcal{MS}_n$. Then $|R_n| = O(n^3)$.*

Proof. Let P be a circle packing for T whose carrier is the open unit disk $\mathbb{D}$, provided by [36]. The main properties of P used in our proof are

(i) two vertices of T are joined with an edge if and only if the corresponding circles are tangent, and

(ii) there are no accumulation points of circles of P inside $\mathbb{D}$.

For a vertex u of T, let x_u denote the corresponding circle of P. Assume without loss of generality that x_o is centered at the origin of the plane.

Assume that $|R_n| = \omega(n^3)$ contrary to our claim. Let R'_n be the set of vertices of R incident with an edge in R_n. Then $|R'_n| > |R_n| = \omega(n^3)$.

For any $u \in R'_n$ Lemma 11.9 yields a connected subgraph G_u of T of at most $2n + 1$ vertices containing u and surrounding o; indeed, G_u can be obtained from any outer-interface M witnessing the fact that $u \in R'_n$ by possibly adding the edge of u lying in ∂M in case u does not lie on M.

Let P_u denote the union of the circles of P corresponding to G_u. We claim that the area $area(P_u)$ covered by P_u is at least r/n^2 for some constant $r = r(P)$. Indeed, P_u is the union of at most $2n+1$ circles, and its diameter is greater than the diameter of x_o, and so at least one of its circles must have diameter of order at least $1/n$, hence area of order at least $1/n^2$.

For every n, pick a subset R''_n of R'_n such that any two vertices of R''_n lie at distance at least $4n + 2$ along R, and therefore in T since R is a geodesic, and $|R''_n| = \omega(n^2)$. Such a choice is possible because $R'_n = \omega(n^3)$. By the same argument, we can assume moreover that any vertex of R''_n is at graph-distance at least $4n + 2$ from any vertex of R''_{n-1}.

Note that for any two distinct elements $u, v \in R''_n$, the subgraphs G_u, G_v defined above are vertex disjoint: this is because, we chose u, v to have distance at least $4n + 2$ in T, and each of G_u, G_v has at most $2n + 1$ vertices and is connected. Moreover, recall that each P_u has area of order at least $1/n^2$. Combining these two facts we obtain $\sum_{u \in R''_n} area(P_u) = \omega(1)$, a contradiction since $area(\mathbb{D})$ is finite. □

Proof of Theorem 11.2. We repeat the arguments of the proof of Theorem 11.1, replacing Lemma 11.6 by Lemma 11.10. □

In the case of recurrent triangulations the theorem of He & Schramm states that T is the contacts graph of a circle packing whose carrier is the plane $\mathbb{R}^2$ [36]. Let P be such a circle packing. We will prove the analogue of Lemma 11.10 for recurrent triangulations of an open disk such that the radii of the circles of P are bounded from above. This in turn implies that $p_{\mathbb{C}} \leq 1/2$ for such triangulations by repeating the proof of Theorem 11.2.

Lemma 11.11. *Let T be a triangulation of an open disk which is recurrent and has bounded vertex degree. Assume that*

$$\text{for some (and hence every) circle packing } P \text{ of } T, \text{ the radius of every disk in } P \text{ is bounded from above by some constant } M. \tag{37}$$

Let R be a geodetic ray in T starting at any $o \in V(G)$, and let R_n be the set of edges of R contained in some outer-interface of $\mathcal{MS}_n$. Then $|R_n| = O(n^5)$.

Proof. We will follow the proof of Lemma 11.10. Assume that $|R_n| = \omega(n^5)$ contrary to our claim. Recall the definitions of P_u, G_u and R'_n, and let R''_n be defined as in the proof of Lemma 11.10 with the additional property $\infty > |R''_n| = \omega(n^4)$. This is possible because $|R_n| = \omega(n^5)$. In the proof of Lemma 11.10 we utilised the finite area of $\mathbb{D}$ to derive a contradiction. However the area of the plane is infinite. For this reason we will construct a family of bounded domains (D_n) with the property that P_u is contained in D_n for every $u \in R''_n$.

Let u_n be the vertex of R''_n that attains the greatest graph distance from o. We claim that $G_n := G_{u_n}$ contains a cycle that surrounds o. Indeed, assuming that G_n does not contain any such cycle, we obtain that o lies in G_n. Moreover, any graph G_u separates o from infinity, and for any two distinct elements $u, v \in R''_n$, the subgraphs G_u, G_v are vertex disjoint, as mentioned in the proof of Lemma 11.10. Thus for any other $u \in R''_n$, G_u has to contain some vertex v of R such that $d(v, o) > d(u_n, o)$, which is a contradiction. Hence G_n contains a cycle C_n that surrounds o.

Let D_n be the domain bounded by that cycle. Arguing as above we can immediately see that each P_u for $u \in R''_n \setminus \{u_n\}$ lies in D_n. Moreover, C_n contains at most $2n$ edges by Lemma 11.9. Every edge of T has length at most $2M$ in P by our assumption, therefore, the length of C_n (as a curve in $\mathbb{R}^2$) is at most $4Mn$.

As in the proof of Lemma 11.10 if $u \in R''_n$, then some circle of P_u has area of order at least $1/n^2$. Hence we obtain $\sum_{u \in R''_n \setminus \{u_n\}} area(P_u) = \omega(n^2)$, since $|R''_n \setminus \{u_n\}| = \omega(n^4)$. Using the standard isoperimetric inequality of the plane we derive

$$\sum_{u \in R''_n \setminus \{u_n\}} 4\pi area(P_u) \leq 4\pi area(D_n) \leq (4Mn)^2.$$

We have obtained a contradiction. □

Using an idea of Grimmett & Li [32], we can slighly improve our results to obtain the strict inequality $p_c \leq p_{\mathbb{C}} < 1/2$ instead of $p_c \leq p_{\mathbb{C}} \leq 1/2$ in all above results. Indeed, it is not hard to see that for any bounded degree triangulation of an open disk T, $\sigma_k(o) \leq 3 \cdot 2^{d-1}(2^d - 2)^{\lfloor n/d \rfloor}$, where d is the maximum degree of T. This comes from the fact that for every vertex u and any edge e incident to u the number of d-step self avoiding walks starting from u that do not traverse e is at most $2^d - 2$. Hence $p_c \leq p_{\mathbb{C}} < 1/2$ as claimed.

11.3 Site percolation

A well-known remark of Grimmett & Stacey [41, §7.4] transforms any upper bound on $p_c(G)$ into an upper bound on $\dot{p}_c$ via the formula $\dot{p}_c \leq 1 - (1 - p_c)^d$

whenever G has maximum degree d. But in our case we can do better: for the triangulations for which we proved $p_{\mathbb{C}} \leq 1/2$ in the previous section we can also prove $\dot{p}_c \leq \frac{1}{d-1}$. For this, instead of working with the dual T^* we work directly with the primal T. We adapt (36) into
$\mathbb{E}_{1-p}(P(u)) \leq \sum_{k=0}^{\infty} d \cdot (d-1)^{k-1}(1-p)^k < \infty$, which yields an analogue of Corollary 11.5 for $p > 1-1/d$. We then proceed as in the proof of Theorem 11.1.

12 Alternating signs of Taylor coefficients

In Section 4.2.1 we proved that the functions $f_m(t) := \mathbb{P}_t(|C(o)| \geq m)$ and $p_m(t) := \mathbb{P}_t(|C(o)| = m)$ are analytic, and even more, they can be extended into entire functions. Thus p_m is uniquely determined by its Maclaurin coefficients. We remark that most 'macroscopic' functions of percolation theory, e.g. χ and θ, are uniquely determined by the sequence $\{p_m\}_{m\in\mathbb{N}}$, and hence by their Maclaurin coefficients. It is rather hopeless to try to determine all these coefficients for any particular percolation model, but perhaps it is less hopeless to e.g. compare two models by comparing the corresponding Maclaurin coefficients.

Motivated by such thoughts we wondered what can be said about those coefficients in general. In this section we determine the signs of the Maclaurin coefficients of f_m and p_m, which turn out not to depend on the model, and deduce that they are alternating. In fact this remains valid in any non trivial percolation model and we do need to impose any transitivity assumption. We let V be a countably infinite set, and μ any function defined on the set $E := V^2$ of pairs of elements of V such that $\sum_{y\in V} \mu(xy) < \infty$ for every $x \in V$, and use this data to obtain a percolation model as defined in Section 2. However, for ease of notation we will assume that $\sum_{y\in V} \mu(xy) = 1$ for every $x \in V$, as in Section 4.2.1. (Some readers may prefer to think of V as the vertex set of a countable connected graph, with μ supported on its edge set E.)

We call an entire function *alternating*, if its Maclaurin coefficients are all real and their signs are alternating. To be more precise, if the Maclaurin series of f is $\sum c_i x^i$, with $c_i \in \mathbb{R}$, we say that f is alternating if $sgn(c_i) = (-1)^{i+\epsilon}$, for some $\epsilon \in \{0,1\}$. Here, the *sign* $sgn(c)$ of a real number $c \neq 0$ is defined as $c/|c|$. With a slight abuse of notation, we allow $sgn(0)$ to take any of the values 1 or -1. For example, any constant real function is allowed as an alternating function.

More generally, we say that f is *alternating at a point* $r \in \mathbb{R}$, if the Taylor coefficients c_i of $f(z)$ at $z = r$ satisfy $sgn(c_i) = (-1)^{i+\epsilon}$.

For an analytic function f, we let $f[k] := \frac{f^{(k)}(0)}{k!}, k \geq 0$ denote the kth Maclaurin coefficient of f. More generally, let $f[k](r)$ denote the kth Taylor coefficient of f at r.

Theorem 12.1. *The (entire extension of the) function f_m is alternating, with $sgn(f_m[k]) = (-1)^{m+1+k}$.*

Since $p_m = f_m - f_{m+1}$, this immediately implies that p_m is alternating too, with $sgn(p_m[k]) = (-1)^{m+1+k}$.

We will prove Theorem 12.1 by induction, and to do so we will prove the following refinement of our statement. Let F, A be non-empty subsets of V, such that F is a finite subset of A. Any percolation instance $\omega \in \{0,1\}^E$ can be restricted to define a random graph A_ω on A by only keeping the edges that are occupied and have both end-vertices in A. By a straightforward extension of Theorem 4.8, we can prove that the function $\mathbb{P}_t(|\cup_{g\in F} C_{A,g}| \geq m)$, where $C_{A,g}$ denotes the component of vertex g in A_ω, admits an entire extension, which we will denote by f_m. Our aim is to prove that f_m is alternating for every $m \geq |F|$, with $sgn(f_m[k]) = (-1)^{m+|F|+k}$. The special case where $F = \{o\}$ and $A = V$ then yields Theorem 12.1.

We will prove this using the following formula:

$$\begin{aligned} f_m(t) = \mathbb{P}_t(|N_{A\setminus F}(g_1)| \geq m - |F|)+ \\ \sum_{n=0}^{m-|F|-1} \sum_{L\in B_n} \mathbb{P}_t(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| \geq m-1)\mathbb{P}_t(N_{A\setminus F}(g_1) = L), \end{aligned} \tag{38}$$

where g_1 is a fixed but arbitrary element of F, and B_n is the set of all possible subsets of size n of the (deterministic) neighbourhood of g_1 in $A \setminus F$, and $S_L := \big(F \setminus \{g_1\}\big) \cup L$.

The fact that this formula holds (for all $t \in \mathbb{R}_+$) is easy to check: we consider all possible neighbourhoods L of g_1 in $A \setminus F$ in our percolation instance, and compute the probability of the event $|\cup_{g\in F} C_{A,g}| \geq m$ defining f_m conditioning on L, except that we bulk all L with $|L| \geq m - |F|$ into the first summand of the right hand side.

We claim moreover that the functions involved in the right hand side admit entire extensions, and that these extensions still satisfy (38) for every $z \in \mathbb{C}$.

Indeed, the first summand can expressed as a sum of simpler functions via the formula

$$\mathbb{P}_t(|N_{A\setminus F}(g_1)| \geq m - |F|) = 1 - \textstyle\sum_{n=0}^{m-|F|-1} \sum_{L\in B_n} \mathbb{P}_t(N_{A\setminus F}(g_1) = L). \tag{39}$$

By Corollary 4.7 all functions of the form $\mathbb{P}_t(N_{A\setminus F}(g_1) = L)$ admit entire extensions and

$$\textstyle\sum_{L\in B_n} |\mathbb{P}_t(N_{A\setminus F}(g_1) = L)| \leq e^{2M} \sum_{L\in B_n} \mathbb{P}_M(N_{A\setminus F}(g_1) = L) < \infty \tag{40}$$

for every $M > 0$ and every $z \in D(0, M)$. Applying the Weierstrass M-test and Weierstrass' Theorem 15.1 as usual we deduce that $\sum_{L\in B_n} \mathbb{P}_t(N_{A\setminus F}(g_1) = L)$ admits an entire extension, and hence so does $\mathbb{P}_t(|N_{A\setminus F}(g_1)| \geq m-|F|)$ by (39).

For the second summand of (38) we observe as above that all functions $\mathbb{P}_t$ involved admit entire extensions and thus it suffices to verify once again the assumptions of the Weierstrass M-test for the series taken when summing over $L \in B_n$. To upper bound $\mathbb{P}_t(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| \geq m-1)$ we will use the identity

$$\mathbb{P}_t(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| \geq m - 1) = 1 - \sum_{j=1}^{m-2} \mathbb{P}_t(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| = j).$$

Using the estimates of Lemma 4.4 and a simple triangle inequality we deduce, for the corresponding entire extensions, that

$$|\mathbb{P}_z(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| \geq m - 1)| \leq 1 + \sum_{j=1}^{m-2} e^{2Mj} P_M(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| = j)$$

for every $M > 0$ and every $z \in D(0, M)$. We can further upper bound $|\mathbb{P}_z(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| \geq m-1)|$ by $1 + (m-2)e^{2Mm}$, because obviously $P_M(|\cup_{g\in S_L} C_{A\setminus\{g_1\},g}| = j) \leq 1$. Combining this with (40) we deduce that the assumptions of the M-test are verified.

This proves that the right hand side of (38) admits an entire extension as claimed. Since this extension coincides with f_m on $\mathbb{R}_+$ as mentioned above, it must coincide with $f_m(z)$ on all of $\mathbb{C}$ by the uniqueness principle since $f_m(z)$ is entire.

Our inductive proof of Theorem 12.1 is based on the observation that all these functions $\mathbb{P}_t$ involved in (38) are alternating themselves, and the following basic fact that products of alternating functions are alternating.

Lemma 12.2. *If f, g are entire alternating functions then fg is also alternating, and $sgn(fg[k]) = (-1)^k sgn(f[k]) sgn(g[k])$.*

Proof. This is an easy combinatorial excersise, using the well-known fact that the Taylor series of a product of two analytic functions coincides with the product of the Taylor series of the two functions at any point of the intersection of their domains of definition. □

We now prove that the entire extensions of the functions of the form $\mathbb{P}_t(N_{A\setminus F}(g_1) = L)$ appearing in (38) are alternating.

Lemma 12.3. *Let L, X be non-empty subsets of V, such that L is a finite subset of X. Then for every $o \in V$, the entire extension f of $\mathbb{P}_t(N_X(o) = L)$ is alternating, with $sgn(f[k]) = (-1)^{|L|+k}$.*

Proof. By definition, our function satisfies the following formula:

$$\begin{aligned} f(z) := \mathbb{P}_z(N_X(o) = L) = \prod_{s\in X\setminus L} e^{-z\mu(os^{-1})} \prod_{s\in L} \left(1 - e^{-z\mu(os^{-1})}\right) = \\ e^{-z\sum_{s\in X\setminus L}\mu(os^{-1})} \prod_{s\in L} \left(1 - e^{-z\mu(os^{-1})}\right). \end{aligned} \tag{41}$$

Since the function $e^{-z\nu}$ is alternating for every real constant ν, the latter expression is a product of $|L| + 1$ alternating functions. Thus the result follows from Lemma 12.2. Indeed, the leftmost factor has its odd Maclaurin coefficients positive, while each of the $|L|$ other factors has its even coefficients positive. □

Next, we prove that the first summand of (38) is also alternating.

Lemma 12.4. *Let F, X be non-empty subsets of V, such that F is a finite subset of X. Then for every $o \in V$, the analytic extension f of $P_z(|N_X(o)| \geq j)$ is alternating for every $j \geq 0$, with $sgn(f[0]) = (-1)^j$.*

Proof. We can rewrite f as

$$f(z) = 1 - \mathbb{P}_z(|N_X(o)| < j) = 1 - \textstyle\sum_{n=0}^{j-1} \sum_{L\in B_n} \mathbb{P}_z(N_X(o) = L). \tag{42}$$

Indeed, this formula is easily verified for $z \in \mathbb{R}_+$, and by the arguments used for (38) it holds for every $z \in \mathbb{C}$.

Note that the right hand side involves j sums, each of which is a sum of alternating functions with agreeing signs by Lemma 12.3. However, the signs of each of those j sums have alternating parities, and since we do not know anything about the absolute values of their coefficients this formula is not enough to prove our statement. However, it will be useful below on different grounds.

We start by proving the statement of the lemma for finite X, using a double induction on $|X|$ and j. To begin with, for $j = 0$, f is alternating, with $sgn(f[k]) = (-1)^k$ for every finite X, as it becomes the constant function $f = 1$. Moreover, f is identically 0 and hence alternating for $X = \emptyset$ and every $j \geq 1$, and we can take $sgn(f[k]) = (-1)^{j+k}$ in this case. For the inductive step, suppose the statement is proved for $j \leq k$ and every finite X. Then for $j = k$, we will prove it by induction on $|X| = 1, 2, \ldots$ (remember we already know it for $|X| = 0$). For this, we can pick any element $x \in X$, and rewrite f as follows, by distinguishing between the events of the edge ox being absent or present:

$$\begin{aligned} &f(z) = P_z(|N_X(o)| \geq j) = e^{-z\mu(ox)} P_z(|N_{X\setminus x}(o)| \geq j) + \\ &(1 - e^{-z\mu(ox)}) P_z(|N_{X\setminus x}(o)| \geq j-1). \end{aligned} \tag{43}$$

(Again, we repeat the arguments used (38) to establish this in all of $\mathbb{C}$.) By our induction hypothesis, both P_z functions involved are alternating; the sign of the kth Maclaurin coefficient of the first one is $(-1)^{j+k}$, while for the second one it is $(-1)^{j-1+k}$. By Lemma 12.2, each of the two products of (43) is alternating, with the sign of the kth coefficient being $(-1)^{j+k}$.

This completes the induction step, establishing that f is alternating for finite X. For an infinite X we now use an approximation argument. Let $X_1 \subset X_2 \subset \ldots$ be an increasing sequence of finite subsets of X with $\bigcup X_i = X$. We claim that each Maclaurin coefficient of $P_z(|N_X(o)| \geq j)$ is the limit, as $i \to \infty$, of the corresponding Maclaurin coefficient of $P_z(|N_{X_i}(o)| \geq j)$. Since we have already proved the latter functions to be alternating because X_i is finite, this claim implies our statement that f is alternating.

Applying (42) with X replaced by X_i for every $i \in \mathbb{N}$, we have

$$f_i(z) := 1 - P_z(|N_{X_i}(o)| < j) = 1 - \textstyle\sum_{n=0}^{j-1} \sum_{\substack{L \in B_n \\ L \subset X_i}} P_z(N_{X_i}(o) = L) \tag{44}$$

To prove the aforementioned claim about the convergence of Maclaurin coefficients, it suffices to show that f_i converges to f uniformly on some open disk $D(0, M)$, and we next show that this is the case.

Using the explicit formula (41), we have

$$P_z(N_{X_i}(o) = L) = P_z(N_X(o) = L) e^{z \sum_{x \in X \setminus X_i} \mu(ox)}$$

whenever $L \subset X_i$. Hence we obtain

$$\sum_{n=0}^{j-1} \sum_{\substack{L \in B_n \\ L \subset X_i}} P_z(N_{X_i}(o) = L) = e^{z \sum_{x \in X \setminus X_i} \mu(ox)} \sum_{n=0}^{j-1} \sum_{\substack{L \in B_n \\ L \subset X_i}} P_z(N_X(o) = L).$$

Pick some $M > 0$, and note that as $i \to \infty$, the last factor $e^{z \sum_{x \in X \setminus X_i} \mu(ox)}$ approaches the constant 1 function uniformly on $D(0, M)$ because

$$|e^{z \sum_{x \in X \setminus X_i} \mu(ox)} - 1| \leq e^{M \sum_{x \in X \setminus X_i} \mu(ox)} - 1$$

for every $z \in D(0, M)$ by Lemma 4.5, and the latter quantity converges to 0. Moreover as $i \to \infty$ the sequence $\sum_{n=0}^{j-1} \sum_{\substack{L \in B_n \\ L \subset X_i}} P_z(N_X(o) = L)$ converges to $\sum_{n=0}^{j-1} \sum_{L \in B_n} P_z(N_X(o) = L)$ uniformly on $D(0, M)$, since

$$\sum_{\substack{L \in B_n \\ L \not\subset X_i}} |P_z(N_X(o) = L)| \leq \sum_{\substack{L \in B_n \\ L \not\subset X_i}} e^{2M} P_M(N_X(o) = L)$$

for every $z \in D(0, M)$ by Lemma 4.4, and the latter sum converges to 0. Therefore f_i converges to f uniformly on $D(0, M)$ as desired. □

We now have all the ingredients needed for Theorem 12.1:

Proof of Theorem 12.1. We work with the more general function $f_m(t) = \mathbb{P}_t(|\cup_{g \in F} C_{A,g}| \geq m)$ as discussed after the statement of Theorem 12.1, and proceed by induction on m. The statement is trivial for $m \leq |F|$, since f_m is the constant function 1 in this case, and we are allowed to consider $sgn(0)$ to be 1 or -1. For the induction step, supposing we have proved the statement for $m < j$, we can obtain it for $m = j$ using (38); we repeat it here for convenience:

$$\begin{aligned} f_m(z) = P_z(|N_{A \setminus F}(g_1)| \geq m - |F|) + \\ \sum_{n=0}^{m-|F|-1} \sum_{L \in B_n} P_z(| \cup_{g \in S_L} C_{A \setminus \{g_1\}, g}| \geq m - 1) P_z(N_{A \setminus F}(g_1) = L), \end{aligned} \tag{45}$$

The first summand is alternating by Lemma 12.4, while we can prove each summand of the form $P_z(| \cup_{g \in S_L} C_{A \setminus \{g_1\}, g}| \geq m - 1) P_z(N_{A \setminus F}(g_1) = L)$ appearing in the second summand to be alternating by combining Lemma 12.2 with our induction hypothesis and Lemma 12.3 (here we used the fact that $|S_L| = |F| + |L| - 1 < m - 1$ since $|L| \leq m - |F| - 1$ in order to be allowed to apply the induction hypothesis). Moreover, it is straightforward to check that these results also imply that the kth Maclaurin coefficient of any of those summands is $(-1)^{m+|F|+k}$. Since the kth Maclaurin coefficient of f_m is the sum of the corresponding coefficients of these finitely many summands, this completes the proof that f_m is alternating, with $sgn(f_m[k]) = (-1)^{m+|F|+k}$.

□

We just proved that f_m and p_m are alternating at 0. Using this we can prove the same for z on the negative real axis.

Corollary 12.5. *The functions f_m and p_m are alternating at every $r \in \mathbb{R}_{\leq 0}$, with $sgn(f_m[k](r)) = sgn(p_m[k](r)) = (-1)^{m+k+1}$.*

Proof. It suffices to prove the statement for f_m, since we can then deduce it for p_m using again the fact that $p_m = f_m - f_{m+1}$.

Since f_m is an entire function, so is it's nth derivative $f_m^{(n)}(z)$, and therefore the radius of convergence of the Maclaurin expansion of $f_m^{(n)}(z)$ is infinite. Thus we can determine the sign of $sgn(f_m^{(n)}[k](r))$ by using the Maclaurin expansion of $f_m^{(n)}(z)$. The latter can be immediately obtained using the Maclaurin expansion of f_m, and we have $sgn(f_m^{(n)}[k]) = sgn(f_m[k + n])$, which by Theorem 12.1 equals $(-1)^{m+1+k+n}$. Evaluating the Maclaurin expansion of $f_m^{(n)}$ at $r < 0$ we see that all terms of that expansion have sign $(-1)^{m+n+1}$, and so $sgn(f_m^{(n)}(r)) = (-1)^{m+n+1}$. Since $sgn(f_m[k](r)) = sgn(f_m^{(k)}(r))$ by the definition of the Taylor expansion, our claim follows. $\square$

We finish this section with a related fact about the zeros of our functions.

Theorem 12.6. *The functions p_m and f_m have a zero of order at least $m-1$ at $z=0$ for every $m>1$.*

Proof. We first prove the statement for p_m. Note that any connected graph with m vertices has at least $m-1$ edges. Hence using the explicit formulas

$$p_m(z) = \textstyle\sum_{S \in G_m} P_z(C(o) = S), \tag{46}$$

where G_m denotes the set of connected graphs on m vertices in V, and

$$P_z(C(o) = S) = \prod_{e \in \partial S} e^{-z\mu(e)} \prod_{e \in E(S)} \left(1 - e^{-z\mu(e)}\right),$$

we see that the summands of p_m have a zero of order at least $m-1$ at $z=0$, because each factor of the form $1 - e^{-z\mu(e)}$ contributes a zero of order 1 and $|E(S)| \geq m-1$. By Theorem 4.8 the partial sums in (46) converge uniformly on an open neighbourhood of 0 to p_m, which implies that p_m satisfies the desired property.

Combining this with the formula $p_m = f_m - f_{m+1}$, we can now easily deduce that f_m too has a zero of order at least $m-1$ at $z=0$. Indeed, by Corollary 12.5 the kth Maclaurin coefficient of f_m and $-f_{m+1}$ have the same sign $(-1)^{m+1+k}$. Hence if any of the first $m-1$ Maclaurin coefficients of f_m or $-f_{m+1}$ is non-zero then so is the corresponding coefficient of p_m, contradicting what we just proved. $\square$

13 The negative percolation threshold

In Section 4.3 we proved that the susceptibility χ is an analytic function of the parameter below the percolation threshold p_c or t_c for all transitive models. This means that $\chi(t)$ admits an extension into a holomorphic function in some domain D of $\mathbb{C}$ containing the interval $(0, p_c)$ or $(0, t_c)$. It would be interesting to come up with a definition that determines this D uniquely, and makes it maximal in some sense. Motivated by this quest, we introduce in this section a 'negative threshold' $t_c^- \in \mathbb{R}_{<0}$, at which the boundary of such a D would have to cross the negative real axis. From now on we will be working with a transitive long-range model as defined in Section 2, but the discussion can be repeated for nearest-neighbour models as well.

The standard percolation threshold t_c is typically defined as $\sup\{t \mid \theta(t) = 0\}$. Natural alternative definitions of t_c can be given by considering the finiteness of the susceptibility χ, i.e. as $\sup\{t \mid \chi(t) < \infty\}$, or in terms of the exponential decay of the cluster size as $\sup\{t \mid \exists c < 1 : p_m(t) \leq c^m \ \forall m \in \mathbb{N}\}$. For a while it was an open problem whether these three thresholds coinside, which was settled by the papers [4, 2] (we discussed in Section 3.2 about how these results generalise to long-range models).

When trying to define the negative threshold t_c^- we are faced with similar difficulties, some of which we are able to overcome below. Perhaps the most natural definition is the following. Since we know (Theorem 4.10) that $\chi(t)$ admits an analytic extension into a domain containing the real interval $[0, t_c)$, we can let I be the largest real interval that contains $[0, t_c)$ and is contained in the domain of an analytic extension of $\chi(t) : \mathbb{R}_+ \to \mathbb{R}_+$, and let $t_c^- = t_A \in \mathbb{R}_- \cup \{-\infty\}$ be the leftmost point of I.

Alternative definitions can be given based on the concrete analytic extension of χ that we constructed with Theorem 4.10: recall that we used the fact that, for $t \in \mathbb{R}_+$, we have $\chi(t) = \sum_m m p_m$. Alternatively, we could have used the formula $\chi(t) = \sum_m f_m$. This motivates the following definitions.

Definition 13.1. *We define* $t_1 := \inf\{r < 0 \mid \lim_{m\to\infty} p_m(r) = 0\}$, $t_2 := \inf\{r < 0 \mid \sum_{m=1}^{\infty} m|p_m(r)| < \infty\}$, $t_3 := \inf\{r < 0 \mid \lim_{m\to\infty} f_m(r) = 0\}$ *and* $t_4 := \inf\{r < 0 \mid \sum_{m=1}^{\infty} |f_m(r)| < \infty\}$.

Moreover, given the important role of the exponential decay of p_m in this paper, it is also natural to define

$$t_5 := \inf\{r < 0 \mid \exists c < 1 : |p_m(r)| \leq c^m \ \forall m \in \mathbb{N}\}.$$

We remark that since $sgn(f_m[k](r))) = (-1)^{m+k+1}$ and $sgn(p_m[k](r))) = (-1)^{m+k+1}$ when $r < 0$ by the results of Section 12, we see that $|f_m(r)|$ and $|p_m(r)|$ are decreasing functions of r for every $m \geq 1$.

We will show that all these values t_i coincide (Theorem 13.3). A key role in our proof will be played by the Hadamard three circles theorem (Theorem 15.3). In order to use it, we first prove that the supremum of both $|f_m|$ and $|p_m|$ over the closed disk $D(0, M)$ is attained at $z = -M$.

Lemma 13.2. *Let f be an alternating function. Then for every $M > 0$*

$$\sup_{z \in D(0,M)} |f(z)| = |f(-M)|.$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be the Taylor expansion of f. Then

$$\left|\sum_{k=0}^{\infty} c_k z^k\right| = \left|\sum_{k=0}^{\infty} (-1)^k c_k (-z)^k\right| \leq \sum_{k=0}^{\infty} |(-1)^k c_k| M^k.$$

Note that the sign of $(-1)^k c_k$ is the same for every k, since $sgn(c_k) = (-1)^{k+\varepsilon}$ for some $\varepsilon \in \{0, 1\}$. Hence,

$$\sum_{k=0}^{\infty} |(-1)^k c_k| M^k = \left|\sum_{k=0}^{\infty} (-1)^k c_k M^k\right| = \left|\sum_{k=0}^{\infty} c_k (-M)^k\right| = |f(-M)|.$$

Thus, f is maximised at $z = -M$. $\square$

Theorem 13.3. *With the above notation we have $t_1 = t_2 = t_3 = t_4 = t_5$.*

Proof. We will show that $t_1 = t_5$, from which the remaining equalities follow easily. It is immediate from the definitions that $t_1 \leq t_5$. To show that $t_1 \geq t_5$, pick $r, r_2 \in \mathbb{R}$ with $t_1 < r_2 < r < 0$. By Theorem 4.8 we have $|P_m(z)| \leq$

$e^{2mM}P_m(M)$ for every $M > 0$ and $z \in D(0,M)$. Therefore, since $P_m(M)$ decays exponentially in m for every $0 < M < t_c$ [4, 7], we can choose $r_1 < 0$ with $|P_m(r_1)| \leq ke^{-lm}$ for some $k, l > 0$ (for this argument we can do without the results of [4, 7]; instead, we can use the fact that $P_m(M)$ decays exponentially in m for $0 < M < 1$, which can be proved by comparison with a subcritical Galton-Watson tree). Pick such an $r_1 < 0$ with $r_1 > r$. Using the Hadamard three circles theorem (Theorem 15.3) and Lemma 13.2, we have

$$|P_m(r)| \leq |P_m(r_1)|^{c_1}|P_m(r_2)|^{c_2},$$

where $c_1 = \dfrac{log|r_2| - log|r|}{log|r_2| - log|r_1|}$ and $c_2 = \dfrac{log|r| - log|r_1|}{log|r_2| - log|r_1|}$. Note that both c_1, c_2 are positive. Since $|P_m(r_2)|$ converges to 0 by the definition of t_1, it follows that $|P_m(r_2)|^{c_2}$ is bounded above by some constant $c > 0$. Moreover, by the choice of r_1, $|P_m(r_1)|^{c_1}$ decays exponentially in m. Hence so does $|P_m(r)|$. This proves that $t_1 = t_5$.

Obviously $t_1 \leq t_2$ and $t_3 \leq t_4$. Using the identity $P_m = f_m - f_{m+1}$ we see that P_m converges to 0 whenever f_m does. This shows that $t_1 \leq t_3$. Also, assuming that $|P_m(r)|$ decays exponentially in m for $t_1 < r < 0$ we obtain that $\sum_{m=1}^{\infty} m|P_m(r)| < \infty$. Hence $t_1 \geq t_2$. Moreover, we have $f_m(r) = \sum_{i=m}^{\infty} P_m(r)$: to see this, note that the functions f_m and $\sum_{i=m}^{\infty} P_m(z)$ coincide on the positive real line. Besides, the exponential decay of $|P_m(r)|$ combined with Lemma 13.2 and the fact that P_m is alternating by the results of Section 12, implies that $\sum_{i=m}^{\infty} P_m(z)$ is continuous on $D(0,M)$ and analytic on its interior. Since f_m is entire, the two functions coincide on $D(0,M)$. Therefore, $|f_m(r)|$ decays exponentially in m for $t_1 < r < 0$ and the series $\sum_{m=1}^{\infty} |f_m(r)|$ converges, which implies that $t_4 \leq t_1$. □

We thus let $t_\chi := t_i$ be our second candidate for the definition of t_c^-. There is one case where we can actually compute t_χ: for the Poisson branching process (which is not one of our percolation models, but our definitions extend to it canonically), we have $t_\chi = W(1/e)$, where W denotes the Lambert function. This implies that for appropriately parametrised percolation on the d-regular tree T_d, we have $\lim_{d\to\infty} t_\chi(T_d) = W(1/e)$[7].

It is natural to ask whether $t_\chi = t_A$, but it turns out that this is not the case: for percolation on the 1-way infinite path, as well as for the Poisson branching process, we have found out that $t_A = -\infty$ although t_χ is finite. Since these two models are the least and the most percolative examples, it might be that $t_A = -\infty$ always holds, and t_χ is the 'right' definition of the negative threshold.

14 Appendix: On the number of lattice animals of a given size

Let T_d denote the infinite d-regular tree, and let $\mathcal{S}_n$ denote the number of subtrees of T_d with n vertices containing a fixed vertex $o \in V(T_d)$. We claim that

$$\mathcal{S}_n < c_d \left(\tfrac{(d-1)^{(d-1)}}{(d-2)^{(d-2)}}\right)^n, \tag{47}$$

[7]The proofs of these and the following facts will be given in the second author's PhD thesis (in preparation).

where c_d is a constant depending on d but not on n.

This can be proved using the following idea due to Kesten [38, Lemma 5.1]. Consider bond percolation on T_d with parameter $p = 1/d-1$ (the critical value). The probability that the cluster C of the root has exactly n vertices is of course at most 1. This probability can be explicitly computed as

$$\mathbb{P}(|C| = n) = \mathcal{S}_n p^{n-1}(1-p)^{(d-2)n+2},$$

since if $|C| = n$ then $|E(C)|n - 1$ and $|\partial C| = (d-2)n + 2$ (the latter can be proved by induction on n). Substituting p by $1/d - 1$ we arrive at (47) by elementary manipulations.

Using (47) we can also upper bound the number of subtrees of any d-regular graph:

Corollary 14.1. *For every graph G with maximum degree d, and any vertex $o \in V(G)$, the number of subtrees of G with n vertices containing o is at most*

$$c_d \left(\frac{(d-1)^{(d-1)}}{(d-2)^{(d-2)}} \right)^n < c_d((d-1)e)^n$$

where c_d is a universal constant depending on d only.

Proof. We may assume without loss of generality that G is d-regular, for otherwise we can attach an appropriate infinite tree to each vertex of degree less than d to raise all degrees to exactly d.

Since G is d-regular, its universal cover is (isomorphic to) T_d, so let $p : T_d \to G$ be a covering map. Fix a preimage o' of o under p. Then every subtree of G containing o lifts uniquely to a subtree of T_d containing o', and distinct subtrees of G lift to distinct subtrees of T_d. This means that the number of subtrees of G containing o is at most the corresponding number for T_d, which is less than $c_d \left(\frac{(d-1)^{(d-1)}}{(d-2)^{(d-2)}} \right)^n$ by (47). We can rewrite the fraction in the parenthesis as

$$(d-1)(\frac{d-1}{d-2})^{(d-2)} = (d-1)(1 + \frac{1}{d-2})^{(d-2)} < (d-1)e$$

to complete our proof.

□

Remark 1: Corollary 14.1 implies that the number of n-vertex induced connected subgraphs of G containing a fixed vertex, called (site) lattice animals in the statistical mechanics literature, or polyominoes in combinatorics, is upper-bounded by the same expression, since every such graph has at least one spanning tree, and no two distinct induced subgraphs share a spanning tree. In particular, we deduce that the growth rate of the number of site lattice animals of any graph of maximum degree d is at most $(d-1)e$. In the special case where G is the $\mathbb{Z}^d$ lattice this upper bound was proved in [11] with different arguments.

Remark 2: The number $\mathcal{S}_n$ is known exactly: it is $\frac{d((d-1)n)!}{(n-1)!((d-2)n+2)!}$.[8] This can be proved using analytic combinatorics. One can also arrive at (47) using Stirling's formula to approximate the factorials in the latter expression.

[8]We thank Stephan Wagner for acquainting us with this formula.

15 Appendix: complex analysis basics

In this appendix we list some classical facts in complex analysis used throughout the paper. They can be found in standard textbooks like [1]. The first two provide the standard technique for showing that a sum of analytic functions is analytic, a technique we employ many times throughout the paper.

Theorem 15.1. ***(Weierstrass Theorem)*** *Let f_n be a sequence of analytic functions defined on an open subset Ω of the plane, which converges uniformly on the compact subsets of Ω to a function f. Then f is analytic on Ω. Moreover, f_n' converges uniformly on the compact subsets of Ω to f'.*

Theorem 15.2. ***(Weierstrass M-test)*** *Let f_n be a sequence of complex-valued functions defined on a subset Ω of the plane and assume that there exist positive numbers M_n with $|f_n(z)| \leq M_n$ for every $z \in \Omega$, and $\sum_n M_n < \infty$. Then $\sum_n f_n$ converges uniformly on Ω.*

The following is only used in Section 13, when we discuss the negative percolation threshold.

Theorem 15.3. ***(Hadamard's three circles theorem)*** *Let $f(z)$ be an analytic function on the annulus $r_1 \leq |z| \leq r_2$. Let $M(r) = \sup\{|f(re^{it})|, t \in \mathbb{R}\}$ be the supremum of $|f(z)|$ over the circle of radius r. Then for every $r \in (r_1, r_2)$*

$$M(r) \leq M(r_1)^R M(r_2)^{R'},$$

where $R = R(r_1, r, r_2) = \dfrac{log r_2 - log r}{log r_2 - log r_1}$ and $R' = R'(r_1, r, r_2) = \dfrac{log r - log r_1}{log r_2 - log r_1}$.

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